Edge-Colorings without Small Substructures

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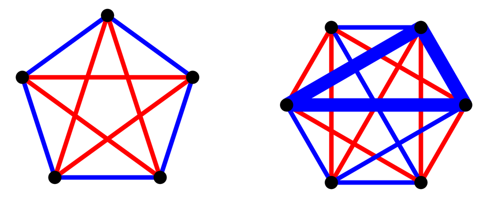
Abstract

Ramsey Theory is a branch of discrete mathematics studying unavoidable patterns in "large enough" systems. We study the minimum number of colors needed in a proper edge-coloring of a complete graph on n vertices to avoid certain small substructures. We give new bounds when avoiding two disjoint color-isomorphic triangles and when avoiding a path on four edges with two colors.

Introduction

We focus on edge-colorings of the complete graph K_n on n vertices. For example, what is the minimum n such that every red-blue coloring of K_n contains a monochromatic triangle? Any red-blue coloring of K_6 will have a monochromatic triangle, but K_5 may not.



We are interested in the number of colors required to avoid such substructures.

Definition 1: A **color pattern** H is an edge-labelled graph. An edge-colored graph G contains H if there is a copy of H in G where edges of the same label receive the same color.

The above graph on the left avoids , but the graph on the right does contain . A **proper edge-coloring** is a coloring avoiding .

Definition 2: Let $\mathcal{F}$ be a set of color patterns. We define $R(n, \mathcal{F})$ to be the minimum number of colors needed to edge-color K_n while avoiding every color pattern in $\mathcal{F}$.

The above example shows $R(5, \ ) \leq 2$ and $R(6, \ ) \geq 3$.

This formalism expresses various Ramsey problems. We investigate $R(n, \{\ ,\ \ \})$ and $R(n, \{\ ,\ \})$.

Avoiding Two Disjoint Color-Isomorphic Triangles

Conlon and Tyomkyn [1] introduced the problem of determining the minimum number of colors needed to properly edge-color K_n with no k disjoint color-isomorphic copies of a fixed graph H. In our language, the simplest nontrivial case is $R(n, \{\ ,\ \ \})$. The first forbidden color pattern restricts us to proper edge-colorings; the second forbids *disjoint* triangles with the same colors. Our starting point is the following result.

Theorem 1 (Conlon-Tyomkyn, 2021): Let $\mathcal{F} = \{\ ,\ \ \}$. Then

$$R(2k+1, \mathcal{F}) = 2k+1 \quad \text{and} \quad R(2k, \mathcal{F}) \in \{2k-1, 2k, 2k+1\}.$$

Proof. For $2k+1$ vertices, each color class is a matching and so can have at most k edges. Since there are $\binom{2k+1}{2} = (2k+1)k$ edges in total, there must be at least $2k+1$ colors, so $R(2k+1, \{\ ,\ \ \}) \geq 2k+1$. A similar counting argument for $2k$ vertices yields $R(2k, \{\ ,\ \ \}) \geq 2k-1$.

We give a geometric coloring of K_{2k+1} with $2k+1$ colors. Space the $2k+1$ vertices equally along a circle. Parallel edges are given the same color, as in the figure.

Color Class 1 / Color Class 2 / Color Class 3 / Color Class 4

Color Class 5 / Color Class 6 / Color Class 7 / Combined

The above coloring is proper and uses $2k+1$ colors. It can be shown that because $2k+1$ is odd, this coloring does not have disjoint color-isomorphic triangles. We can also remove a vertex from the above coloring of K_{2k+1} to get a coloring of K_{2k}. This gives

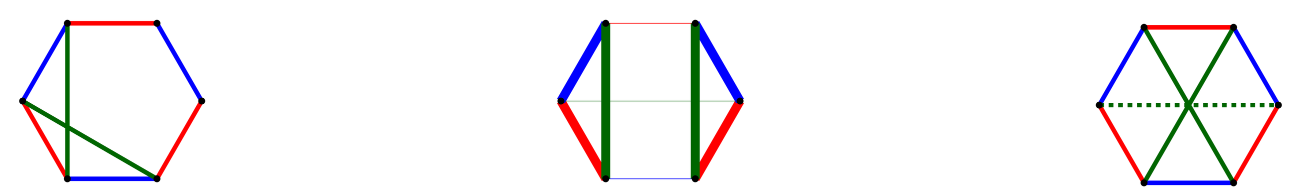
$$R(2k+1, \{\ ,\ \ \}) \leq 2k+1 \quad \text{and} \quad R(2k, \{\ ,\ \ \}) \leq 2k+1. \qquad \square$$

The preceding theorem completely solves $R(n, \{\ ,\ \ \})$ for odd n. We improve on this for some small even cases.

Theorem 2 (EEJKT): Let $\mathcal{F} = \{\ ,\ \ \}$. Then

$$R(6, \mathcal{F}) = 7 \quad \text{and} \quad R(8, \mathcal{F}) \in \{8, 9\}.$$

Proof. We only prove the weaker $R(6, \mathcal{F}) \geq 6$ here. Suppose, for the sake of contradiction, that K_6 can be properly colored with 5 colors while avoiding disjoint color-isomorphic triangles. In order to have a proper coloring of K_6, each color class must have 3 edges and be a perfect matching. Then the union of 2 color classes must form a collection of even cycles. Thus every 2 color classes must form a hexagon. The only way to assign another perfect matching to this hexagon is to use the diameters:



0 diameters - no third edge / 1 diameters - color-isomorphic triangles / 2 diameters - requires a third diameter

Thus any 3 perfect matchings in K_6 form a hexagon plus 3 diameters, leaving no space for more perfect matchings. Hence K_6 cannot be colored with 5 colors. $\square$

Avoiding an Alternating Path of Length 4

Here we consider $R(n, \{\ ,\ \})$ where we are restricted to proper edge-colorings that avoid paths on 4 edges with alternating colors. Rosta [3] gave a coloring that satisfies these conditions that uses $n-1$ colors for $n = 2^k$ vertices. Keevash and Sudokov [2] later demonstrated this can only be done when $n = 2^k$.

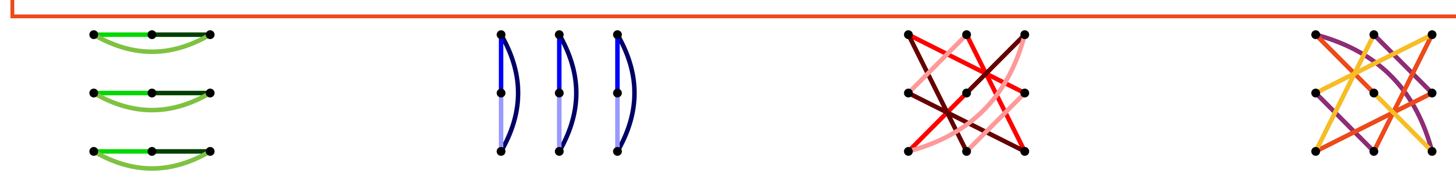
Theorem 3 (Keevash-Sudakov, 2005):

$$R(n, \{\ ,\ \}) = n-1 \text{ if and only if } n = 2^k.$$

If $n \neq 2^k$, the previous theorem gives us 2 basic bounds. Let 2^k be the first power of 2 greater than n. Then, $R(n, \{\ ,\ \}) \geq n$ and $R(n, \{\ ,\ \}) \leq 2^k - 1$. We can improve this upper bound in various cases.

Theorem 4 (EEJKT):

$$R(9, \{\ ,\ \}) \leq 12.$$



Colors 1, 2, 3 / Colors 4, 5, 6 / Colors 7, 8, 9 / Colors 10, 11, 12

One can check that the union of any two of these colors avoids an alternating path. With the following theorem, we can use any prior colorings to build colorings for larger graphs. For example, the bound $R(18, \{\ ,\ \}) \leq 31$ can be improved to 25.

Theorem 5 (EEJKT): Let $\mathcal{F} = \{\ ,\ \}$. Then

$$R(ab, \mathcal{F}) + 1 \leq (R(a, \mathcal{F}) + 1)(R(b, \mathcal{F}) + 1).$$

Proof. Let $[n] = \{1, \dots, n\}$, and let C_1 and C_2 be sets of colors of size $R(a, \mathcal{F}) + 1$ and $R(b, \mathcal{F}) + 1$, resp. Let

$$\chi_1 : [a]^2 \to C_1 \quad \text{and} \quad \chi_2 : [b]^2 \to C_2$$

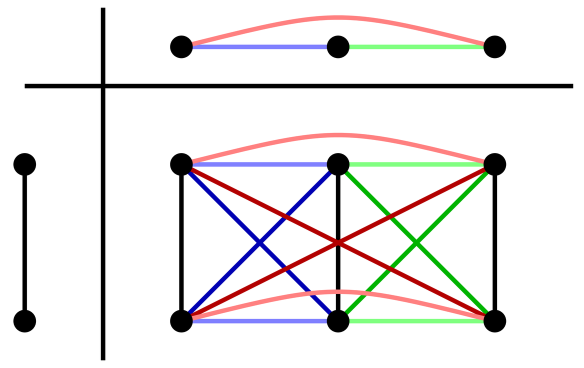
be colorings of the complete graph on a and b vertices, resp., which avoid and , where we also color loops. Define

$$\chi : ([a] \times [b])^2 \to C_1 \times C_2$$

by

$$\chi((i_1, j_1), (i_2, j_2)) = (\chi_1(i_1, i_2), \chi_2(j_1, j_2)).$$

Then χ is a coloring of the complete graph on ab vertices, where we also color loops, using $(R(a)+1)(R(b)+1)$ colors. See the Figure.



Claim 1: χ avoids . Suppose there are two incident edges of the same color:

$$\chi((i_1, j_1), (i_2, j_2)) = \chi((i_1, j_1), (i_3, j_3)).$$

Looking at each coordinate, we have $\chi_1(i_1, i_2) = \chi_1(i_1, i_3)$ and $\chi_2(j_1, j_2) = \chi_2(j_1, j_3)$. Since χ_1 and χ_2 avoid , these imply $i_2 = i_3$ and $j_2 = j_3$, which means that the initial edges were the same.

Claim 2: χ avoids . Suppose there is such an alternating path starting at (i_1, j_1) and ending at (i_5, j_5). Looking at the first coordinate, since χ_1 has no such alternating path, we have $i_1 = i_5$. Similarly, $j_1 = j_5$. Thus $(i_1, j_1) = (i_5, j_5)$ which is a contradiction.

The coloring χ colors K_{ab} with $(R(a, \mathcal{F}) + 1)(R(b, \mathcal{F}) + 1)$ colors and is monochromatic cherry and alternating color path free. Since we color the loops with all the same color, we have $R(ab, \mathcal{F}) \leq (R(a, \mathcal{F}) + 1)(R(b, \mathcal{F}) + 1) - 1$. $\square$

Open Questions

1. Improve the bounds on $R(n, \{\ ,\ \})$ for $n \neq 2^k$.
2. Determine $R(n, \{\ , A_k\})$ where A_k is an alternating path of length $k > 4$.
3. Determine $R(2k, \{\ ,\ \ \})$ for more values of k.

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References

[1] D. Conlon, M. Tyomkyn. Repeated patterns in proper colourings. *SIAM J. Discrete Math.* (2021).
[2] P. Keevash, B. Sudakov. On a hypergraph Turán problem of Frankl. *Combinatorica* (2005).
[3] V. Rosta. Note on Gy. Elekes's conjectures concerning unavoidable patterns in proper colorings. *Electron. J. Combin.* (2000).