Edge-Colorings without Small Substructures

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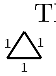
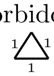
Abstract

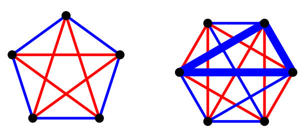
Ramsey Theory is a branch of discrete mathematics studying unavoidable patterns in “large enough” systems. We study the minimum number of colors needed in a proper edge-coloring of a complete graph on n vertices to avoid certain small substructures. When avoiding two disjoint color-isomorphic triangles, we give new bounds for small values and present a related problem. We also give new bounds when avoiding a path on four edges with two colors along with a method to construct new colorings for larger graphs.

1 Introduction

We focus on edge-colorings of the complete graph K_n on n vertices. In order to represent avoiding certain substructures, we introduce the following notation.

Definition 1.1. *A* ***color pattern*** *H is an edge-labelled graph. An edge-colored graph G contains H if there is a copy of H in G where edges of the same label receive the same color.*

This notation is convenient to express forbidden structures. For example, the graph on the left avoids  , while the graph on the right contains  .



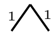
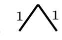
We are interested in the minimum number of colors needed to color graphs with forbidden color patterns. To formalize this, we use the following notation.

Definition 1.2. *Let $\mathcal{F}$ be a set of color patterns. We define $R(n, \mathcal{F})$ to be the minimum number of colors needed to edge-color K_n while avoiding every color pattern in $\mathcal{F}$.*

Proposition 1.3 (Monotonicity)**.** *If $n < m$, then $R(n, \mathcal{F}) \leq R(m, \mathcal{F})$. If $\mathcal{F}' \subset \mathcal{F}$, then $R(n, \mathcal{F}') \leq R(n, \mathcal{F})$.*

Proof. There is a coloring of K_m which uses $R(m, \mathcal{F})$ colors while avoiding all color patterns in $\mathcal{F}$. Delete any $m - n$ vertices from K_m. This results in a coloring of K_n which uses up to $R(m, \mathcal{F})$ colors while still avoiding these same patterns. Hence $R(n, \mathcal{F}) \leq R(m, \mathcal{F})$.

For the second claim, there is a coloring of K_n using $R(n, \mathcal{F})$ colors while avoiding every color pattern in $\mathcal{F}$. This coloring is also a coloring of K_n that avoids all the color patterns of $\mathcal{F}'$. Hence $R(n, \mathcal{F}') \leq R(n, \mathcal{F})$. ☐

This formalism expresses various Ramsey problems. For instance, we can represent the minimum number of colors for a proper edge-coloring of K_n as $R(n, \{$  $\})$. As we are primarily interested in proper colorings, we recall the following simple counting argument which gives a best-possible lower bound on $R(n,$  $)$.

Lemma 1.4. *$R(2k, \{$  $\}) \geq 2k - 1$ and $R(2k + 1, \{$  $\}) \geq 2k + 1$.*

Proof. First, we focus on K_{2k+1}, which has $(2k+1)(k)$ edges. Note that a color class can have at most k edges, since this is the maximum size of a matching in K_{2k+1}. This means $2k+1$ colors are always needed. For K_{2k}, the amount of edges is $(k)(2k-1)$. Again, the maximum number of edges per color is k, so at least $2k-1$ colors are needed. This gives us

$$R(2k+1, \{ \wedge_{(1,1)} \}) \geq 2k+1 \quad \text{and} \quad R(2k, \{ \wedge_{(1,1)} \}) \geq 2k-1. \qquad \square$$

We investigate $\mathcal{F} = \{ \wedge_{(1,1)}, \triangle_{(1,2,3)}\ \triangle_{(1,2,3)} \}$ in Section 2 and $\mathcal{F} = \{ \wedge_{(1,1)}, P_{(1,2,1,2)} \}$ in Section 3. By monotonicity, the lower bounds in Lemma 1.4 are lower bounds for these problems as well. We present further results on these problems.

1.1 Avoiding color-isomorphic disjoint triangles

Conlon and Tyomkyn [2] introduced the problem of determining the minimum number of colors needed to properly edge-color K_n with no k disjoint color-isomorphic copies of a fixed graph H. In our language, the simplest nontrivial case is $R(n, \{ \wedge_{(1,1)}, \triangle_{(1,2,3)}\ \triangle_{(1,2,3)} \})$. The first forbidden color pattern restricts us to proper edge-colorings; the second forbids *disjoint* triangles with the same colors.

Theorem 1.5 (Conlon, Tyomkyn[2]). *Let* $\mathcal{F} = \{ \wedge_{(1,1)}, \triangle_{(1,2,3)}\ \triangle_{(1,2,3)} \}$. *Then*

$$R(2k+1, \mathcal{F}) = 2k+1 \quad \text{and} \quad R(2k, \mathcal{F}) \in \{2k-1, 2k, 2k+1\}.$$

The above theorem completely solves the problem on complete graphs with an odd number of vertices, but there is still some improvement left in the even cases. We solve the problem on K_6.

Proposition 1.6. $R(6, \{ \wedge_{(1,1)}, \triangle_{(1,2,3)}\ \triangle_{(1,2,3)} \}) = 7$.

Additionally, we investigate how adding additional forbidden structures changes the bounds.

Theorem 1.7. *If* $k \geq 2$, $R(2k, \{ \wedge_{(1,1)}, \triangle_{(1,2,3)}\ \triangle_{(1,2,3)}, \square_{(1,2,1,2)} \}) = 2k+1$.

This previous theorem helps us improve the bounds on K_8.

Proposition 1.8. $R(8, \{ \wedge_{(1,1)}, \triangle_{(1,2,3)}\ \triangle_{(1,2,3)} \}) \geq 8$.

1.2 Avoiding paths on four edges with two colors

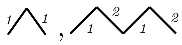
Here we consider $R(n, \{ \wedge_{(1,1)}, P_{(1,2,1,2)} \})$ where we are restricted to proper edge-colorings that avoid paths on 4 edges with alternating colors. Disproving a conjecture of Elekes, Rosta [4] gave such a coloring that uses $n-1$ colors when $n = 2^k$. In relation to another Ramsey problem, Axenovich [1] proved $R(n, \{ \wedge_{(1,1)}, P_{(1,2,1,2)}, \square_{(1,2,1,2)} \}) \leq 2n^{1+\frac{c}{\sqrt{\log n}}}$, where c is a positive constant, with the upper bound seeming very hard to improve in contrast to Theorem 1.7. Keevash and Sudokov [3] later demonstrated that Rosta's result can only be done when $n = 2^k$.

Theorem 1.9 ([3]). $R(n, \{ \wedge_{(1,1)}, P_{(1,2,1,2)} \}) = n-1$ *if and only if* $n = 2^k$ *for some* $k \in \mathbb{N}$.

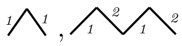
In other words, $R(n, \{ \wedge_{(1,1)}, P_{(1,2,1,2)} \}) \geq n$ unless $n = 2^k$. We improve this lower bound by 1 wherever possible.

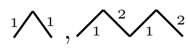
Theorem 1.10. *$R(n, \{$  $\}) = n$ if and only if $n = 2^k - 1$ for some $k \in \mathbb{N}$.*

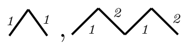
We improve the lower bound further for some small cases.

Theorem 1.11. *$R(5, \{$  $\}) = 7$.*

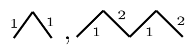
We also give a coloring distinct from Rosta's for 12 vertices, based off of an affine plane of order 3.

Theorem 1.12. *$R(9, \{$  $\}) = 12$.*

With the following theorem, we can use any prior colorings to build colorings for larger graphs. For example, using the previous coloring, the bound $R(18, \{$  $\}) \leq 31$ can be improved to 25.

Theorem 1.13. *Let $\mathcal{F} = \{$  $\}$. Then*

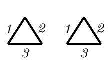
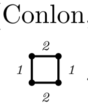
$$R(ab, \mathcal{F}) + 1 \leq (R(a, \mathcal{F}) + 1)(R(b, \mathcal{F}) + 1).$$

Below we have a table which summarizes our bounds on $R(n, \{$  $\})$ for small n.

n	Lower Bound	Upper Bound
2	1	1
3	3	3
4	3	3
5	7	7
6	7	7
7	7	7
8	7	7
9	12	12
10	12	15
11	12	15
12	13	15
13	14	15
14	15	15
15	15	15
16	15	15
17	18	25
18	19	31
19	20	31

2 Avoiding color-isomorphic disjoint triangles

First, we recall a proof of Theorem 1.5 for completeness. To do this, we first prove the following lemma.

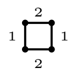
Lemma 2.1 (Conlon, Tyomkyn[2])**.** *There exists a proper coloring of K_{2k+1} with $2k$ colors that avoids*  *and*  *.*

Proof. Label all the vertices of K_{2k+1} from 1 to $2k+1$. For the edges between vertices a and b, assign the color $a + b \mod 2k+1$. This coloring is proper. Let vertices x, p, q be distinct in K_n. This means $x + p \not\equiv x + q \mod n$, so the edge between x and p has a different color than the edge between x and q.

This coloring also avoids color-isomorphic disjoint triangles. Assume there are disjoint triangles formed by x_1, x_2, x_3 and y_1, y_2, y_3, with the following color-isomorphism.

$$\begin{aligned} x_1 + x_2 &\equiv y_1 + y_2 \mod n \\ x_2 + x_3 &\equiv y_2 + y_3 \mod n \\ x_1 + x_3 &\equiv y_1 + y_3 \mod n \end{aligned}$$

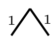
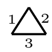
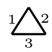
Adding these equations and dividing by 2 gives $x_1 + x_2 + x_3 \equiv y_1 + y_2 + y_3 \mod n$, and subtracting each previous equation gives $x_1 \equiv y_1$, $x_2 \equiv y_2$, and $x_3 \equiv y_3$. Hence these triangles are both the same triangle. Thus these triangles are not disjoint, so no disjoint color isomorphic triangles are present in this coloring.

Lastly, this coloring avoids  . Suppose there are 4 vertices x_1, x_2, x_3, and x_4 that form the previous structure. Let the edge between x_1, x_2 and the edge between x_3, x_4 have the color c_1 while the edge between x_2, x_3 and the edge between x_1, x_4 have the color c_2. This gives us the equations

$$\begin{aligned} x_1 + x_2 \equiv x_3 + x_4 \equiv c_1 \mod 2k+1 \\ x_1 + x_4 \equiv x_2 + x_3 \equiv c_2 \mod 2k+1 \end{aligned}$$

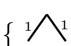
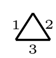
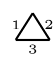
Summing these and simplifying gives us $x_1 \equiv x_3 \mod 2k+1$. Hence these vertices were actually the same and this isn't a 4 cycle. □

Now, using this lemma, we can prove Theroem 1.5.

Proof of Theorem 1.5. Let $\mathcal{F} = \{$  ,   $\}$. By Lemma 1.4 combined with monotonicity, we get the following lower bounds on the problem.

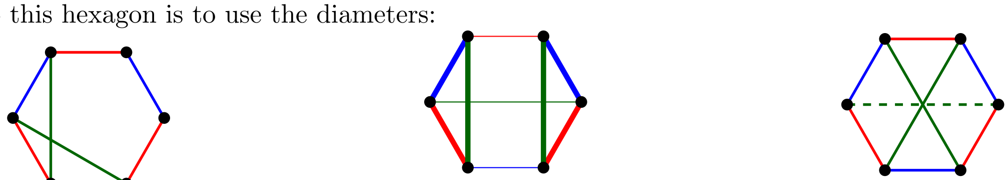
$$R(2k+1, \mathcal{F}) \geq 2k+1 \quad \text{and} \quad R(2k, \mathcal{F}) \geq 2k-1.$$

Using Lemma 2.1, K_{2k+1} can be colored in $2k+1$ colors while avoiding these structures. This gives us $2k+1 \leq R(2k+1, \mathcal{F}) \leq 2k+1$. Since R is monotone, we have $R(2k, \mathcal{F}) \leq R(2k+1, \mathcal{F}) = 2k+1$. Thus $2k-1 \leq R(2k, \mathcal{F}) \leq R(2k+1, \mathcal{F}) = 2k+1$. □

As previously stated, the preceding theorem completely solves $R(n, \{$  ,   $\})$ for odd n. We improve on this for K_6.

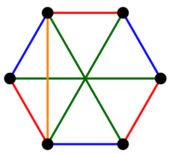
Proof of Proposition 1.6. Suppose, for the sake of contradiction, that K_6 can be properly colored with 6 colors while avoiding disjoint color-isomorphic triangles. In order to have a proper coloring of K_6, there are at least 3 color classes with 3 edges and are a perfect matching. If two color classes are perfect matchings, then their union is a 2 regular graph, and is thus a collection of cycles. In particular, each cycle must be even, since odd cycles cannot be proper edge-colored with 2 colors.

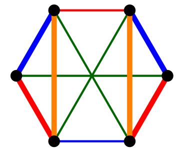
Thus 2 of the perfect matching color classes must form a hexagon. The only way to assign another perfect matching to this hexagon is to use the diameters:

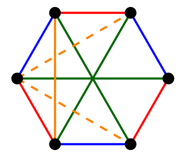


0 diameters - no third edge   1 diameters - color-isomorphic triangles   2 diameters - requires a third diameter

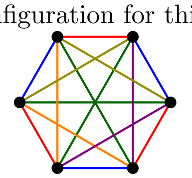
Thus these 3 perfect matchings in K_6 form a hexagon plus 3 diameters, leaving no space for more perfect matchings. Hence the remaining color classes must all use 2 edges. In order to fit these edges in the existing hexagon, both edges of the same color must be incident to a common red or blue edge.

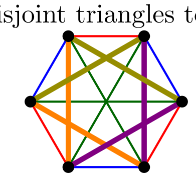

Arbitrary orange edge


Causes color-isomorphic triangles


Possible orange choices

This means the remaining 3 colors must have edges that are incident to a shared red or blue edge. There is only 1 possible configuration for this, but it causes color-isomorphic disjoint triangles to form.


Remaining colors


Disjoint color isomorphic triangles

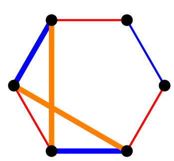
Thus K_6 cannot be colored in 6 colors, so $R(6, \{ {}^{1}\!\wedge^{1}, \overset{1\;2}{\underset{3}{\triangle}}\ \overset{1\;2}{\underset{3}{\triangle}} \}) \geq 7$. By Theorem 1.5, this gives $R(6, \{ {}^{1}\!\wedge^{1}, \overset{1\;2}{\underset{3}{\triangle}}\ \overset{1\;2}{\underset{3}{\triangle}} \}) = 7$. □

Next, we consider what happens if we also forbid ${}_1\overset{2}{\underset{2}{\square}}{}_1$, which we will call a two color square.

Proof of Theorem 1.7. Suppose K_{2k} can be colored with $2k$ colors while avoiding the above structure. K_{2k} has $k(2k-1)$ edges. For the sake of contradiction, suppose there are at most $(k-1)$ perfect matchings, which color k edges each. Then, the rest of the colors contain at most $k-1$ edges. This gives us $(k-1)k + (2k-(k-1))(k-1) = (k-1)(2k+1) = 2k^2 - k - 1 < k(2k-1)$. This means 1 edge was not colored, so we need at least k perfect matchings.

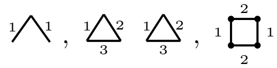
Now, consider 2 perfect matching colors. As we previously showed, they form even cycles. In particular, they can't form a 2 color square for this problem, so they must form cycles of size 6 or larger. Note that for cycles of size $l \geq 6$, the amount of diagonals that immediately form a triangle is l. Hence in the entire graph, there are $2k$ such diagonals. Since we have used up 2 colors, there are $2k-2$ colors left. This means there is some pair of diagonals with the same color.

Since each of these diagonals create a triangle, if these diagonals aren't both incident to some shared edge, then the triangles will be disjoint. In order for these diagonals to form triangles that aren't disjoint, they will instead form a 2 color square as below.

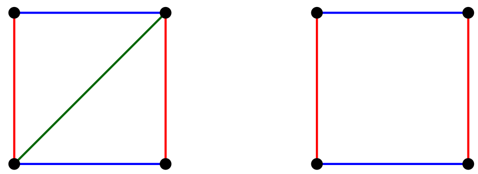


Thus in all possible cases, a forbidden structure appears. This gives $R(2k, \{ {}^{1}\!\wedge^{1}, \overset{1\;2}{\underset{3}{\triangle}}\ \overset{1\;2}{\underset{3}{\triangle}}, {}_1\overset{2}{\underset{2}{\square}}{}_1 \}) \geq 2k+1$. From Lemma 2.1, we have a coloring on K_{2k+1} which avoids all the above structures. Thus we have $2k+1 \leq R(2k, \{ {}^{1}\!\wedge^{1}, \overset{1\;2}{\underset{3}{\triangle}}\ \overset{1\;2}{\underset{3}{\triangle}}, {}_1\overset{2}{\underset{2}{\square}}{}_1 \}) \leq R(2k+1, \{ {}^{1}\!\wedge^{1}, \overset{1\;2}{\underset{3}{\triangle}}\ \overset{1\;2}{\underset{3}{\triangle}}, {}_1\overset{2}{\underset{2}{\square}}{}_1 \}) \leq 2k+1$. Hence $R(2k, \{ {}^{1}\!\wedge^{1}, \overset{1\;2}{\underset{3}{\triangle}}\ \overset{1\;2}{\underset{3}{\triangle}}, {}_1\overset{2}{\underset{2}{\square}}{}_1 \}) = 2k+1$. □

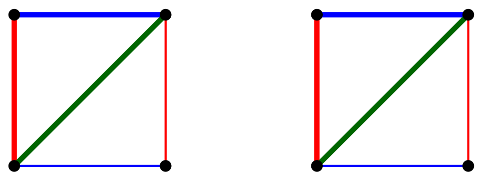
We use the previous theorem to improve the lower bound for K_8.

Proof of Proposition 1.8. Suppose K_8 can be colored with 7 colors. This means that every color class will be a perfect matching. Note that $R(8, \{$  $\}) = 9$. Hence, if a coloring of K_8 avoids 2 disjoint color isomorphic triangles but only uses 7 colors, it must have a two color square. Let red and blue be the two color classes that form a two color square. Since they are a perfect matching, the union must form 2 of these squares, since their union should be a collection of even cycles.

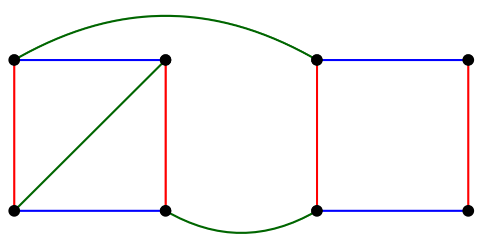
Now, pick any diagonal in one of these squares and give it an arbitrary color. Without loss of generality, we pick this diagonal in the left square and color it green.

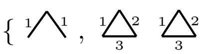
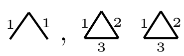


This forms 2 red blue green triangles which are not disjoint. Note this means the diagonals in the other square cannot be green. Otherwise, another red blue green triangle will form that is disjoint to these ones.



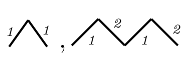
Thus, all green edges incident to a vertex on the right square must also be incident to some vertex on the left square. However, there are only 2 available vertices on the left square, so at most 2 more green edges can be drawn.

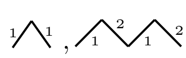
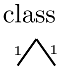


However, all colors are perfect matchings. This is a contradiction, so $R(8, \{$  $\}) \neq 7$. Hence $R(8, \{$  $\}) \geq 8$. ☐

3 Avoiding paths on four edges with two colors

The proof of Theorem 1.10 depends on the parity of the number of vertices, so break the statement into two cases.

Proposition 3.1. *$R(2k-1, \{$  $\}) = 2k-1$ if and only if $2k-1 = 2^m - 1$ for some $m \in \mathbb{N}$.*

Proof. Recall Theorem 1.9 when n is even, $R(n, \{$  $\}) = n-1$ if and only if n is a power of two. Consider K_{n-1} where $n-1$ is odd. Assume that K_{n-1} has been colored using exactly $n-1$ colors. Then, the average amount of edges in each color class is $\frac{\binom{n-1}{2}}{n-1} = \frac{n-2}{2}$, and since any color class having more than $\frac{n-2}{2}$ edges would cause the formation of a , all color classes must have exactly $\frac{n-2}{2}$ edges. The only way for this to be possible while all color classes avoid a $^{1}\wedge^{1}$ is if each color class is a near perfect matching - that is, each color class pairs up all but one vertex via an edge.

For a given near perfect matching, call the singular vertex on the graph which is not incident to an edge of the matching its "lonely" vertex. Note that no two near perfect matchings can have the same lonely vertex, because if that were the case, the vertex would not be adjacent to a sufficient number of edges without the formation of a $^{1}\wedge^{1}$. Thus, there is a one-to-one correspondence between a given color class and its lonely vertex.

Consider adding a new vertex, a, to K_{n-1} to form K_n without using any additional colors but while still avoiding $^{1}\wedge^{1}$ and $\wedge_{1}^{2}\wedge_{1}^{2}$. We can easily avoid the formation of any $^{1}\wedge^{1}$ by assigning the edge connecting a to a given vertex to be the color under which that vertex is lonely.

Now, we set out to prove that this graph avoids $\wedge_{1}^{2}\wedge_{1}^{2}$. Assume, for the sake of contradiction, that the addition of a has formed $\wedge_{1}^{2}\wedge_{1}^{2}$. Since there had been no $\wedge_{1}^{2}\wedge_{1}^{2}$ prior, a must be one of the vertices of the path. Then, there are three cases up to the symmetry of the path: either a is the first vertex of the path, the second, or the third.

Denote the colors of the edges of the $\wedge_{1}^{2}\wedge_{1}^{2}$ as "red" and "blue." If a is the first vertex of the path, let the vertex which is adjacent to a and which forms the alternating path be b, and let the edge between a and b be red. There is a blue edge between b and some vertex c, a red edge between c and some vertex d, and a blue edge between d and some vertex e. Since b, c, d and e were vertices contained within the original K_{n-1}, they must not have formed $\wedge_{1}^{2}\wedge_{1}^{2}$ prior to the introduction of a. Thus, to avoid $\wedge_{1}^{2}\wedge_{1}^{2}$, there must be a red edge between b and e. However, b is already adjacent to a red edge, which means that there is a $^{1}\wedge^{1}$, which is a contradiction.

If a is the second vertex of the path, then without loss of generality there is a red edge connecting a and b and a blue edge connecting a and c, a red edge connecting c and d, and a blue edge connecting d and e. The edges cd and de were present in the graph prior to the introduction of a, so they must not have formed $\wedge_{1}^{2}\wedge_{1}^{2}$. The only way for this to be avoided is if there is a red edge connecting e and some vertex f and a blue edge connecting f and c. However, this creates a contradiction, because this again implies the formation of $^{1}\wedge^{1}$ by the introduction of a.

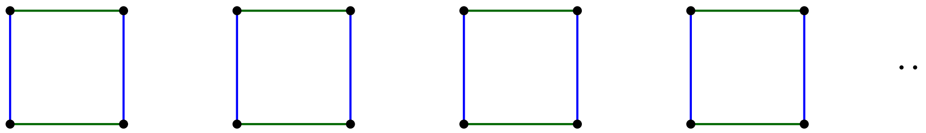
Lastly, a may be the third vertex of the graph. Then, there is a blue edge bc, a red edge ab, a blue edge ad and a red edge de. The edge de was present in the graph prior to the introduction of a, and it must be adjacent to some blue edge ef. If f were not incident to any red edge in the original graph, then it would be the lonely vertex of the red color class. However, this is not the case, because b is the lonely vertex of that color class. Thus, ef must be adjacent to some red edge fg. To avoid an alternating path, there must be a blue edge gd, but again, this causes the formation of $^{1}\wedge^{1}$, which is a contradiction. Thus, our new graph of K_n avoids both $^{1}\wedge^{1}$ and $\wedge_{1}^{2}\wedge_{1}^{2}$.

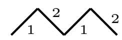
Since K_n has been colored using $n-1$ colors such that $^{1}\wedge^{1}$ and $\wedge_{1}^{2}\wedge_{1}^{2}$ are avoided, n must be a power of two by Theorem 1.9. Thus, if a complete graph of an odd number of vertices has been colored using the same amount of colors as the amount of vertices it has while avoiding $^{1}\wedge^{1}$ and $\wedge_{1}^{2}\wedge_{1}^{2}$, then the number of vertices must be one less than a power of two. □

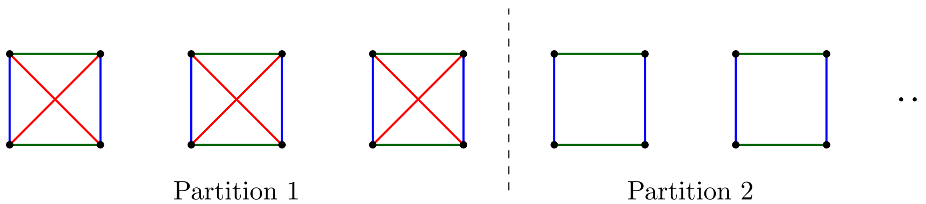
To prove Theorem 1.10 when the number of vertices is even, we use the following reduction lemma.

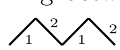
Lemma 3.2. *Let $m \geq 2$. If there exists m disjoint perfect matchings on $2m$ vertices which avoids $\wedge_{1}^{2}\wedge_{1}^{2}$ then m is even and there exists $\frac{m}{2}$ disjoint perfect matchings on m vertices which avoids $\wedge_{1}^{2}\wedge_{1}^{2}$.*

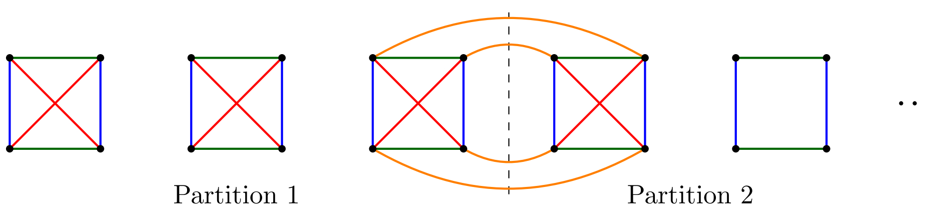
Proof. Fix one of the perfect matchings in K_{2m} (we will color these edges blue below). Then, the union of this perfect matching and any other perfect matching (we will color these new edges green) must form 4 cycles in order to avoid alternating paths. For K_{2m} to be represented as a collection of 4 cycles, this means $2m$ is divisible by 4, or m is even.



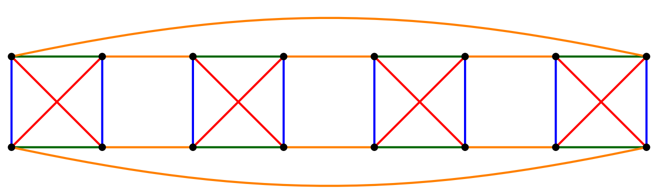
Now, consider if one of the perfect matchings (we will color this matching red) has an edge that is a diagonal in one of these blue green 4 cycles. Then, in order to avoid , the other diagonal in the 4 cycle must be red. Furthermore, we claim that for every blue green 4 cycle, the diagonals will then be red. Suppose for the sake of contradiction that there are at most $2k < m$ red edges that are diagonals in a blue green 4 cycle. Partition the vertices into 2 sets: the first will contain all the vertices incident to one of these $2k$ red edges, while the second will contain all other vertices.



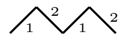
Consider the smaller of the two partitions. This partition has at most m vertices. Pick any of these vertices and call it v. Since there are m perfect matchings, there are m edges leaving v. However, since there are at most $m-1$ other vertices in this partition, there is some edge leaving v going to the other partition. Thus, there is a perfect matching between the partitions (we will color this orange). Both sides have blue and green edges, so to avoid  from appearing, this orange edge forces 3 other orange edges to connect a blue green 4 cycle in partition 1 to a blue green 4 cycle in partition 2. This in turn forces the red edges to appear in partition 2 as diagonals. Hence there were at least $2(k+1)$ red edges that form diagonals. Thus by contradiction, all m red edges form diagonals.

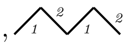


We can reduce the above perfect matchings to K_m by flattening the graph along the blue edges. For example,



turns into

With the exception of the blue edges, each of the previous $m-1$ perfect matchings turned into a perfect matching in K_m that is proper and avoids  . Furthermore, if an edge is used in 2 perfect matchings in K_m, then these perfect matchings, along with the corresponding blue edges, formed cycles with diagonals in K_{2m}. Thus if 2 perfect matchings in K_m share an edge, they share every edge. Furthermore, this means for any perfect matching in K_{2m}, there is only 1 other perfect matching in K_{2m} that reduces to the same matching in K_m. We can then group the $m-1$ perfect matchings in K_{2m} in the following way. If 2 perfect matchings reduce to the same matching in K_m, we pair them. Otherwise, this matching is kept in it's own group. Each of the previous groups gets reduced into a disjoint perfect matching in K_m. Since m is even and we have $m-1$ matchings, there must be at least 1 matching that does not have a pair. However, the remaining $m-2$ matchings could all form pairs. For these pairs, there are at most $\frac{m-2}{2}$ pairs which corresponds to $\frac{m}{2}-1$ perfect matchings in K_m. Combined with the matching without a pair gives us $\frac{m}{2}$ disjoint perfect matchings in K_m. $\square$

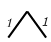
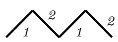
Proposition 3.3. *Let n be an even integer such that $n \neq 2^k$. Then* $R(n, \{$  ,  $\}) \geq n+1$.

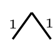
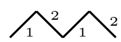
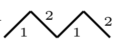
Proof. Consider K_n where n is even and $n \neq 2^k$. Assume that K_n has been colored using exactly n colors. Then, the total amount of edges is $\frac{n(n-1)}{2}$. Suppose there are at most $\frac{n}{2}-1$ color classes that are perfect matchings. The remaining $\frac{n}{2}+1$ color classes have at most $\frac{n}{2}-1$ edges. This means the total amount of edges colored is at most $(\frac{n}{2}-1)(\frac{n}{2})+(\frac{n}{2}+1)(\frac{n}{2}-1) = \frac{n(n-1)}{2}-1 < \frac{n(n-1)}{2}$. Hence there is at least $\frac{n}{2}$ color classes that are perfect matchings.

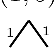
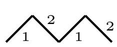
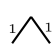
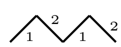
Since n is even but not a power of 2, n can be written as $2^a \cdot b$ where b is odd and $b > 1$. By repeatedly applying Lemma 3.2, we get that b is even giving us a contradiction. Therefore, $n+1$ colors are needed to color K_n.

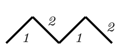
$\square$

Proposition 3.1 and Proposition 3.3 combine to give Theorem 1.10. We now prove the remaining theorems.

Theorem 3.4. $R(5, \{$  ,  $\}) = 7$.

Proof. By Theorem 1.9 $R(8, \{$   $\}) = 7$. By monotonicity, $R(5, \{$ $\}) \leq 7$.

Let G be a K_5 graph and create an arbitrary ordering of $V(G)$ using [5]. Assume that K_5 can be colored with 6 colors. Since $|E(G)| = 10$ and there are 6 color classes, the average size of each color class is $\frac{10}{6}$. Since the K_5 is monochromatic cherry free, each color class has a maximum cardinality of $\lfloor \frac{5}{2} \rfloor = 2$. Therefore, 4 of the color classes must have size 2 and therefore must be almost perfect matchings. Let "1" be the first color class that is an almost perfect matching. Without loss of generality, color the following set of edges "1": $(1,2), (3,4)$. Let "2" be the second color class that is an almost perfect matching. Assume an edge added to "2" is incident to vertex 5. Without loss of generality, $(1,5)$ will be added to "2". The other edge to "2" can be either $(2,3)$ or $(2,4)$. If $(2,3)$ is added to "2", there will be the alternating color path $(5,1,2,3,4)$. If $(2,4)$ is added to "2", there will be the alternating color path $(5,1,2,4,3)$. Therefore, the edges added to "2" must not be incident to vertex 5. Without loss of generality, the edges of "2" will be $(1,3)$ $(2,4)$. Let "3" be the second color class that is an almost perfect matching. Since the K_5 is monochromatic cherry free, vertex 5 must be incident to edges that belong to 4 distinct color classes. Therefore, one edge of color class "3" must be incident to vertex 5. Without loss of generality, assume that $(1,5)$ is in color class "3". Therefore, $(1,5)$ must belong to "3". This however creates the $(5,1,2,3,4)$. Therefore, it can be concluded that $R(5, \{$   $\}) \geq 7$. Thus, $R(5, \{$   $\}) = 7$. $\square$

Theorem 3.5. $R(9, \{$  ,  $\}) \geq 12$.

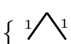
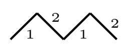
Proof. Let G be a K_9 graph and create an arbitrary ordering of $V(G)$ using [9]. Assume that K_9 can be colored with 11 colors. Since $|E(G)| = 36$ and there are 11 color classes, the average size of each color class is $\frac{36}{11}$. Since the K_9 is monochromatic cherry free, each color class has a maximum cardinality of $\lfloor \frac{9}{2} \rfloor = 4$. Since there are 11 color classes, at least 3 of the color classes must have size 4 and therefore must be almost perfect matchings. Let "1" be the first color class that is an almost perfect matching. Without loss of generality, color the following set of edges "1": $(1,2),(3,4),(5,6),(7,8)$.

Let "2" be the second color class that is an almost perfect matching. Assume that the color class "2" contains an edge that is incident to vertex 9. Without loss of generality, let the edge $(1,9)$ be in "2". Next assume that the color class "2" contains an edge that is incident to vertex 2. Without loss of generality, add $(2,4)$ to "2". This creates an alternating color path $(9,1,2,4,3)$. Therefore, no edges that belong to color class "2" can be incident to vertex 9. Without loss of generality, add $(1,3)$ to color class "2". All vertices except vertex 9 must be incident to an edge belonging to color class "2". Edge $(2,4)$ must then be added to color class "2" since an edge from vertex to 2 to any other vertex that is not already incident to an edge belonging to color class "2" will create an alternating color path. Without loss of generality, let $(5,7)$ and $(6,8)$ belong to color class "2".

Let "3" be the third color class that is an almost perfect matching. Assume that the color class "3" contains an edge that is incident to vertex 9. Without loss of generality, let the edge $(1,9)$ be in "3". Next assume that the color class "3" contains an edge that is incident to vertex 2. Without loss of generality, assume that $(2,3)$ belong to color class "3". This creates an alternating color path $(9,1,2,3,4)$. Therefore, no edges that belong to color class "3" can be incident to vertex 9.

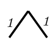
Let $(2,3)$ belong to color class "3". This means that $(1,4)$, $(5,8)$, and $(6,7)$ must belong to "3" as any other possible edges would create a color alternating path. There are 8 more color classes and 24 edges of K_9 that need to be assigned to a color class. Therefore, the average cardinality of the remaining color classes must be $\frac{24}{8}=3$. Thus, 4 of the remaining color classes must have cardinality ≥ 3. Let "4" be a color class with cardinality ≥ 3. Since the K_9 is monochromatic cherry free, vertex 9 must be incident to edges that belong to 8 distinct color classes. Since no edges of color classes "1", "2", or "3" are incident to vertex 9, one edge of color class "4" must be incident to vertex 9. Without loss of generality, assume that $(1,9)$ is in color class "4". No other edges can be assigned to color class "4" without creating a alternating color path so a contradiction has been reached.

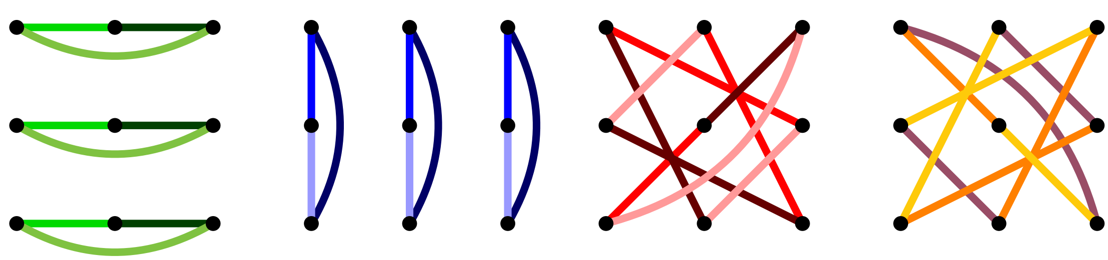
Therefore, $(2,3)$ can not belong to color class "3". Without loss of generality, assume that $(1,7)$ belongs to color class "3". This means that $(3,5)$, $(2,8)$, and $(4,6)$ must belong to "3" as any other possible edges would create an alternating color path. Without loss of generality, assume that $(1,9)$ is in color class "4".

Vertices 1, 2, 3, and 7 can not be incident to edges belonging to color class "4" as this would create an alternating color path. Therefore, vertices 4,5,6, and 8 must be incident to edges belonging to color class "4" since 2 more edges need to be added to the color class. However, $(4,6),(5,6)$ and $(6,8)$ have already been assigned to color classes. Therefore, any edge that is added to "3" will lead to a contradiction. Therefore, K_9 can not be colored with 11 colors. Thus, $R(9,\{$   $\})\geq 12$.

$\square$

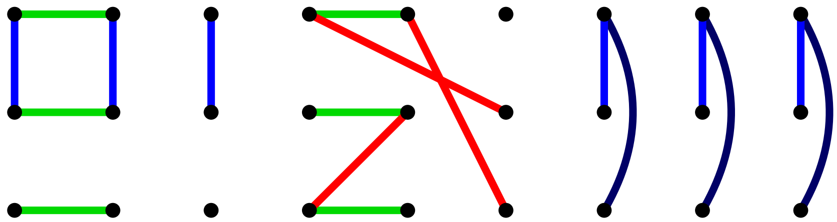
The previous bound is in fact best-possible, as the following example shows.

Theorem 3.6. $R(9,\{$  ,  $\})\leq 12$.

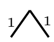
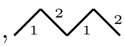


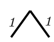
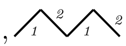
Colors 1, 2, 3 | Colors 4, 5, 6 | Colors 7, 8, 9 | Colors 10, 11, 12

One can check that the union of any two of these colors avoids an alternating path. The union of any two of these colors is isomorphic to one of the following:



Union 1 Union 2 Union 3

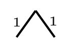
With the following theorem, we can use any prior colorings to build colorings for larger graphs. For example, the bound $R(18, \{$  ,  $\}) \leq 31$ can be improved to 25.

Theorem 3.7. *Let* $\mathcal{F} = \{$  ,  $\}$. *Then*

$$R(ab, \mathcal{F}) + 1 \leq (R(a, \mathcal{F}) + 1)(R(b, \mathcal{F}) + 1).$$

Proof. Let $[n] = \{1, \ldots, n\}$, and let C_1 and C_2 be sets of colors of size $R(a, \mathcal{F})+1$ and $R(b, \mathcal{F})+1$, respectively. Let

$$\chi_1 : [a]^2 \to C_1 \quad \text{and} \quad \chi_2 : [b]^2 \to C_2$$

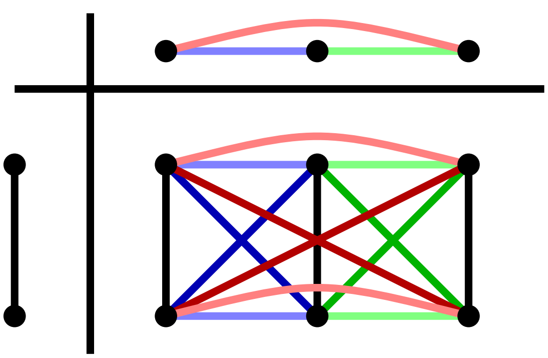
be colorings of the complete graph on a and b vertices, respectively, which avoid  and  , where we also color loops. Define

$$\chi : ([a] \times [b])^2 \to C_1 \times C_2$$

by

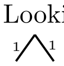
$$\chi((i_1, j_1), (i_2, j_2)) = (\chi_1(i_1, i_2), \chi_2(j_1, j_2)).$$

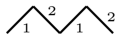
Then χ is a coloring of the complete graph on ab vertices, where we also color loops, using $(R(a)+1)(R(b)+1)$ colors. See the following figure:



Claim 1: χ avoids  . Suppose there are two incident edges of the same color:

$$\chi((i_1, j_1), (i_2, j_2)) = \chi((i_1, j_1), (i_3, j_3)).$$

Looking at each coordinate, we have $\chi_1(i_1, i_2) = \chi_1(i_1, i_3)$ and $\chi_2(j_1, j_2) = \chi_2(j_1, j_3)$. Since χ_1 and χ_2 avoid  , these imply $i_2 = i_3$ and $j_2 = j_3$, which means that the initial edges were the same.

Claim 2: χ avoids . Suppose there is such an alternating path starting at (i_1, j_1) and ending at (i_5, j_5). Looking at the first coordinate, since χ_1 has no such alternating path, we have $i_1 = i_5$. Similarly, $j_1 = j_5$. Thus $(i_1, j_1) = (i_5, j_5)$ which is a contradiction. The coloring χ colors K_{ab} with $(R(a, \mathcal{F})+1)(R(b, \mathcal{F})+1)$ colors and is monochromatic cherry and alternating color path free. Since we color the loops with all the same color, we have $R(ab, \mathcal{F}) \leq (R(a, \mathcal{F}) + 1)(R(b, \mathcal{F}) + 1) - 1$. □

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