Investigations in Combinatorial Game Theory

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Impartial Combinatorial Games

Definition (Impartial Combinatorial Game)

A **game** has

- two players
- take turns making moves
- moves come from a shared set
- last one to move wins

Subtraction Games

Definition (Subtraction Game)

Let S be a set of positive integers.
A **subtraction game** S is a game in which players

- remove tokens from a shared pile
- number of tokens removed is in S
- last one to move wins

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A **subtraction game** S is a game in which players

- remove tokens from a shared pile
- number of tokens removed is in S
- last one to move wins

Ex. $S = \{2, 3\}$

- If the pile has 1 or 0 tokens, the next player loses.
- If the pile has 3 tokens, then the next player wins.

P & N Positions

Definition (P Position)

A **P position** is a position where the *previous* player has a winning strategy

For $S = \{2, 3\}$, a pile with 1 token is a P position.
For $S = \{2, 3\}$, a pile with 5 tokens is a P position.

Definition (N Position)

A **N position** is a position where the *next* player has a winning strategy

For $S = \{2, 3\}$, a pile with 3 tokens is an N position.

P & N Positions

Definition (P Position)

A **P position** is a position where the *previous* player has a winning strategy

Definition (N Position)

A **N position** is a position where the *next* player has a winning strategy

- A position G is a N position *iff* at least one position reachable from G is a P position.
- A position G is a P position *iff* all positions reachable from G are N positions.

Winning strategy: move to a P position.

Determining P and N Positions

Subtraction Game with Set $\{2, 3\}$

Tokens	P/N
0	P
1	P
2	N
3	N
4	N
5	P
6	P
7	N
8	N
9	N
10	P

$(S, T)_k$ Notation

Many classic games can be written in the form $(S, T)_k$

- S: Set of pieces you are allowed to remove from a pile
- T: Set of number of piles you are allowed to play on
- k: Total number of piles in the game

$(S, T)_k$ Notation Example: Subtraction Games

Notation: $(S, \{1\})_1$

- S represents the set of pieces you are allowed to remove.
- $T = \{1\}$: Only allowed to remove from one pile.
- $k = 1$: Only pile in the game.

$(S, T)_k$ Notation Example: Nim

Notation: $(\mathbb{Z}^+, \{1\})_k$

- $S = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, so you are allowed to remove whatever you want from a pile.
- $T = \{1\}$: Only allowed to remove from one pile.
- k can be anything.

P-positions: The addition of the piles in binary without carrying is equal to 0.

$(S, T)_k$ Notation Example: Moore's Nim

Notation: $(\mathbb{Z}^+, \{1, 2, 3, ..., \ell\})_k$

- $S = \mathbb{Z}^+$: allowed to remove whatever you want.
- $T = \{1, 2, 3, ..., \ell\}$: allowed to remove from up to ℓ piles.*
- k can be anything.

*Note: You can remove different numbers of pieces from different piles.

P-positions: The addition of the piles in binary without carrying in base $\ell + 1$ is equal to 0.

$(S, T)_k$ Notation Example: Princess and Roses

Notation: $(\{1\}, \{1, 2\})_k$

- $S = \{1\}$: must remove one piece from a pile.
- $T = \{1, 2\}$: must remove from one or two piles.
- k can be anything.

P-Positions: We have P-positions for number of piles $k \leq 4$.

- 4-pile: Piles are P-positions iff all piles are even or three piles are odd and the smallest is even.

$(\mathbb{Z}^+, \{k\})_{2k}$

Rules for the game	Proposal for P-position	Proving the result
Ex. $(\mathbb{Z}^+, \{2\})_4$ • "4"= # of piles • "$\mathbb{Z}^+$" = subtract any positive integer. • "2" = two piles must be subtracted from each turn.	Three 'a' values and one value of $a + c$. • a is any non-negative integer • $c \geq 0$ $a, a, a, (a + c)$	ex. $3, 4, 5, 7$ $\rightarrow 3, 3, 3, 7$ ex. $3, 3, 5, 5$ $\rightarrow 3, 3, 3, 3$ ex. $4, 4, 4, 2$ $\rightarrow 2, 2, 4, 2$ ex. $4, 4, 4, 7$ $\rightarrow 4, 4, 2, 5$

$(S, \{1, \ldots, k\})_k$

P positions of $(\{2, 3\}, \{1, 2, 3\})_3$:

- all pile sizes $0, 1 \pmod 5$

P and N-positions for subtraction game with $S = \{2, 3\}$:

pile size mod 5	0	1	2	3	4
P or N	P	P	N	N	N

P-position criteria:
Each pile is a P-position in it's own subtraction game

Why This Works: $(S, \{1, \ldots, k\})_k$

1. When in a P-position, any pile cannot be moved to another P-position in its own subtraction game.
2. From an N-position the player can change each of the piles into a P-position of its own subtraction game.
3. Eventually, all of the piles will be P-positions, and will be too small to remove from.

$(S, \{1, \ldots, k\})_{k+1}$ for any subtraction set S

P positions of $(\{2,3\},\{1,2\})_3$:

- all pile sizes $0, 1 \pmod 5$
- all pile sizes $2, 3 \pmod 5$
- all pile sizes $4 \pmod 5$

Nimbers for subtraction game with $S = \{2, 3\}$:

pile size mod 5	0	1	2	3	4
Nimber	0	0	1	1	2

P-position criteria:
All the piles have the same Nimber in their own subtraction game.

Future Work

We also figured out $(\{1\}, T)_3$.

- Extend to $(\{1\}, T)_4$
- P-positions of $(\{1\}, \{1, 2\})_6$
- Relate $(\mathbb{Z}^+, \{1, 2\})_4$ to $(\{1, \ldots, k\}, \{1, 2\})_4$

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Thanks for listening