Investigations in Combinatorial Game Theory

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July 5, 2020

1 Introduction

Impartial combinatorial games (hereafter called 'games') have two players, where each player takes turns making moves. Their moves come from a shared set, and in normal play, the game eventually ends and the last one to move wins. Note that in any game, either the first player or the second player has a winning strategy. It is more convenient to label particular positions of games as first or second player winning. This prompts the standard definition of P- and N-positions.

Definition 1. *A P-position is a position where the previous player has a winning strategy. A position G is a P-position iff all positions reachable from G are N-positions.*

An N-position is a position where the next player has a winning strategy. A position G is a N-position iff at least one position reachable from G is a P-position.

When playing a game, the strategy is to always move into a P-position. If they player can move into a P-position after each round, they will be able to win, as they are the previous player in that round. From a P-position it is only possible to move into an N-position, so the next player will be forced to move into a losing game.

A more refined notion of P- and N-positions are useful in studying some games.

Definition 2. *The Sprague-Grundy function $\mathcal{G}$ is a function from impartial combinatorial games to $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, defined by the MEX rule: for a game G, let $H_1, H_2, \dots$ be the games reachable from G with a single move. We define $\mathcal{G}(G) = mex(\mathcal{G}(H_1), \mathcal{G}(H_2), \dots)$, where $mex(S) = \min(\mathbb{N} \setminus S)$.*

With the notion of the P- and N-positions and the Sprague-Grundy function, we looked at the following types of games.

Definition 3. *We denote by $(S, T)_k$ the following combinatorial game: there are k piles of tokens, each turn consists of choosing $t \in T$ piles and removing $s \in S$ tokens from each pile, possibly removing different numbers of tokens from different piles. We call S the subtraction set.*

Below we have a (partial) list of famous games that fit the $(S,T)_k$ format. The aim of this project was continuing the study of these games more generally. This included writing a program to compute the P-positions of an arbitrary game $(S,T)_k$, and discovering and proving results about particular instances of these games.

Subtraction Games

Subtraction games are $(S,\{1\})_1$, with S an arbitrary subtraction set. Subtraction games with finite subtraction sets are periodic, and the winning strategies can be found via a simple recursion using the two P and N rules above.
There are also subtraction games that have more than one pile and can be denoted as $(S,\{1\})_k$. In these games, the P-positions occur at the positions where the XOR (binary addition without carrying) of the Sprague-Grundy function of the piles is 0.
Nim is a popular example of the multi-pile subtraction games, and it is written as $(\mathbb{Z}^+,\{1\})_k$, where $\mathbb{Z}^+$ is used to denote the set of positive integers. The solution to Nim can be written in a simpler manner than the general multi-pile subtraction game, as the Sprague-Grundy value of a pile with subtraction set $\mathbb{Z}^+$ is simply the size of the pile. Therefore, if the bitwise XOR of the sizes of the piles is 0, it is a P-position.

Moore's Nim

Moore's Nim is a variation of Nim, and it is denoted as $(\mathbb{Z}^+,\{1,...,\ell\})_k$ The players are allowed to remove different numbers of pieces from different piles. The solution to Moore's Nim is very similar to Nim, as the pile sizes are still expanded in binary. However, instead of addition without carrying in base 2, it is performed in base $\ell+1$.

Princess and Roses

Princess and Roses is denoted by $(\{1\},\{1,2\})_k$. We found the solution for up to four piles. Piles are P-positions *iff* the sizes of all four piles are even or three piles are odd and the smallest is even. To expand this solution to less than four piles, one or more of the pile sizes can be set to 0.

2 Three pile generalizations of Princess and Roses

Here we study games of the form $(\{1\},T)_3$ for various choices of T. Most choices of T were easy to analyze; the major exceptions were $T=\{2\}$ and $T=\{2,3\}$, whose analyses were similar.

One pile at a time

This is a subtraction game with three piles. Since the subtraction set is $\{1\}$, it is played the same as a subtraction game with one pile. As we are treating the piles as one big pile, the P-positions are where the sum of the sizes of the three piles are even.

Two piles at a time

Denote the positions of $(\{1\},\{2\})_3$ as (a,b,c), where a, b, and c are the number of tokens in each pile in increasing order, so $a \leq b \leq c$. We first examined cases for $a = 0$, which is equivalent to $(\{1\},\{2\})_2$. We found that $(0,b,c)$ is a P-position if and only if b is even. This generalizes to the following rule.

Theorem 4. *In $(\{1\},\{2\})_3$, let the position be denoted by (a,b,c) with $a \leq b \leq c$. If $c \geq a+b$, then (a,b,c) is a P-position iff a and b are both even.*

We found this by reasoning that if $c \geq a+b$, then whether the position (a,b,c) is P or N is determined by the P and N of the Princess and Roses of (a,b).

However, we also wanted to solve for the P- and N-positions for $c < a+b$. In order to do so, we reduce the 3-pile position (a,b,c) to an ordered pair $(a, c-b)$. For example, the position $(6,7,9)$ is reduced to $(6,2)$. Eventually, we found that further reducing this modulo 4 gave a nice pattern:

(0, 0) is a P-position iff b even	(1, 0) is a P-position iff b even
(0, 1) is a P-position iff b even	(1, 1) is never a P-position
(0, 2) is a P-position iff b even	(1, 2) is a P-position iff b odd
(0, 3) is a P-position iff b odd	(1, 3) is never a P-position
(2, 0) is a P-position iff b even	(3, 0) is a P-position iff b odd
(2, 1) is a P-position iff b odd	(3, 1) is never a P-position
(2, 2) is a P-position iff b even	(3, 2) is a P-position iff b even
(2, 3) is a P-position iff b even	(3, 3) is never a P-position

This can be restated more concisely:

Theorem 5. *In $(\{1\},\{2\})_3$, let the position be denoted by (a,b,c) with $a \leq b \leq c$. If $c < a+b$, then (a,b,c) is a P-position if and only if one of the following conditions are met:*

- *a,b,c are all even,*
- *$a+c-b = 1 \pmod 4$ and b is even,*
- *$a+c-b = 3 \pmod 4$ and b is odd.*

Three piles at a time

This is the same as a subtraction game with one pile, as only the size of the smallest pile matters. Once all the pieces are gone from the smallest pile, the next player will not have any moves, as they must play on exactly three piles. Therefore, the P-positions are where the smallest pile is an even number.

One or two piles at a time

This is simply 3-pile Princess and Roses, so the P-positions are where the parity of all three piles are the same.

Two or three piles at a time

For the $(\{1\},\{2,3\})_3$ game, we were able to find patterns in the P-positions by splitting the values into 2 cases. Similar to the $T = \{2\}$ case, we observed that if $c \geq a + b$, then the game is played like 2-pile Princess and Roses, $(\{1\},\{1,2\})_2$.

Theorem 6. *In $(\{1\},\{2,3\})_3$, let the position be denoted by (a,b,c) with $a \leq b \leq c$. If $c \geq a + b$, then (a,b,c) is a P-position iff a and b are both even.*

When $c < a + b$, the analysis becomes more difficult. Eventually, we arrived at the following theorem:

Theorem 7. *In $(\{1\},\{2,3\})_3$, let the position be denoted by (a,b,c) with $a \leq b \leq c$. If $c < a + b$, then (a,b,c) is a P-position iff one of the following conditions are met:*

- *a is even, b is even, and $a + b + c \not\equiv 3 \pmod 4$,*
- *$a + b \equiv 1 \pmod 2$ and $a + b + c \equiv 1 \pmod 4$.*

It is interesting that we can determine some of the P-positions (when $c \geq a + b$ as in Theorem 4 and 6) without determining all of them.

One or three piles at a time

This is very similar to the subtraction game with one pile, and the solution for the P-piles is the same. The P-positions occur when the sum of the sizes of the piles is even. This is because every move removes an odd number of tokens in total.

Any number of piles at a time

This falls into the category of the $(S,\{1,\ldots,k\})_{k+1}$ game, as one of the sub-cases of that game is $(S,\{1,\ldots,k\})_k$. In the following section we show that games of those sort have all P-positions satisfying each of the piles being a P-Position in their own individual game. So in this case, every pile would have to have an even size to be a P-position.

3 Not all piles at a time

We were interested in changing the subtraction set of Princess and Roses, $(\{1\},\{1,2\})_k$. One way to do this that generalized nicely was $(S,\{1,\ldots,k\})_{k+1}$.

For example, the P-positions of $(\{2,3\},\{1,2\})_3$ are:

- all pile sizes $0,1 \pmod 5$
- all pile sizes $2,3 \pmod 5$
- all pile sizes $4 \pmod 5$

This pattern is explained by the Sprauge-Grundy function on $(S,\{1\})_1$. For example, the $\mathcal{G}$-values of the subtraction game where $S=\{2,3\}$ are

pile size mod 5	0	1	2	3	4
$\mathcal{G}$	0	0	1	1	2

This pattern generalizes into the following theorem:

Theorem 8. *The P-positions of game $(S,\{1,\ldots,k\})_{k+1}$ are when all the pile sizes have the same $\mathcal{G}$-values in a subtraction game with subtraction set S.*

By setting a pile to have size 0, we see that the P-positions of $(S,\{1,\ldots,k\})_k$ are when all the pile sizes are P-positions in their own subtraction games.

Proof. We break the proof into two parts. Recall that the set $\mathcal{P}$ of P-positions of a game is the unique set of positions satisfying: every move from a position in $\mathcal{P}$ is to a position not in $\mathcal{P}$, and for every game not in $\mathcal{P}$ there is a move to a position in $\mathcal{P}$.

1. When the game is in a P-position, the next player must move to an N-position, because they must remove from some, but not all of the piles. Therefore the piles they remove from will have differing $\mathcal{G}$-values from the piles they don't from, since in a subtraction game it is impossible to move from a $\mathcal{G}$-value, to a position with an equivalent $\mathcal{G}$-value.

2. When the game is in an N-position, the next player always has the option of moving into a P-position. In an N-position, at least one $\mathcal{G}$-value will differ from the others, and so the next player can change all of the piles to have the same $\mathcal{G}$-value as the smallest pile, which is possible due to the MEX rule. Since the next player can always make the $\mathcal{G}$-values equal to the smallest $\mathcal{G}$-value that is already present, they can always move to a P-position.

□

4 Exactly k piles at a time

We focused on $(\mathbb{Z}^+, \{2\})_4$, but the results should generalize to any $(\mathbb{Z}^+, \{k\})_{2k}$.

Theorem 9. *The P-positions of $(\mathbb{Z}^+, \{2\})_4$ are when the smallest 3 piles are the same, that is, the piles have sizes a, a, a, and d, for $0 \leq a \leq d$.*

Proof. As in Section 3, we break the proof into two parts. The set $\mathcal{P}$ of P-positions of a game is the unique set of positions satisfying: every move from a position in $\mathcal{P}$ is to a position not in $\mathcal{P}$, and for every game not in $\mathcal{P}$ there is a move to a position in $\mathcal{P}$.

1. When the game is in a P-position, it is of the form (a, a, a, d) with $d \geq a$, and the player must remove from two piles. If the player is to move to another P-position, they must choose the smaller of the two piles they did not play on to be the size of the smallest piles. There are two options for the set of piles to play on: a pile of size a and a pile of size d, or two piles of size a.

   In the first case, either a or d must be changed to a pile of size a, as those are the remaining two pile sizes. This is impossible with the pile of size a, as you must remove a positive number of pieces. So with this move the player must move into an N-position.

   In the second case (the player takes from two piles of size a), the player will have to match it with either a or d, which are the remaining pile sizes. This is not possible, as they player must remove a positive number of pieces.

2. Assume $0 \leq a \leq b \leq c \leq d$, and consider the N-position (a, b, c, d). Since this is an N-position, it is not possible for a, b, c to be equal, as that would make it a P-position. Therefore, $a \neq c$. To turn (a, b, c, d) into a P-position, you can reduce c, d to a. This reduces to (a, a, a, b), and since $b \geq a$, it is a P-position. Thus from any N-position we can move to a P-position.

□

5 Further Questions

There are many interesting questions to be answered about $(S, T)_k$ games. We would like to extend our analysis of $(\{1\}, T)_3$ to four or more piles. Additionally, we would like to answer the classic problem of determining the P-positions of 6-pile Princess and Roses, $(\{1\}, \{1, 2\})_6$. It is likely the answers to these problems will be similar to one ones given above.

On the more exploratory side, given the connection between $(\mathbb{Z}^+, \{1, \ldots, k\})_{k+1}$ and $(S, \{1, \ldots, k\})_{k+1}$ in Section 3, we would be very interested in any relation between $(\mathbb{Z}^+, \{1, 2\})_4$ and $(S, \{1, 2\})_4$. We are more optimistic about such a relationship for $S = \{1, \ldots, k\}$ than for general S.

Acknowledgements

We would like to thank Robert Krueger for his help and guidance, and also the Illinois Geometry Lab and David Frankel for their institutional support.

References

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