New Results Towards the Erdős-Fishburn Problem

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Polymath Jr. 2023 Distinct Distances Group

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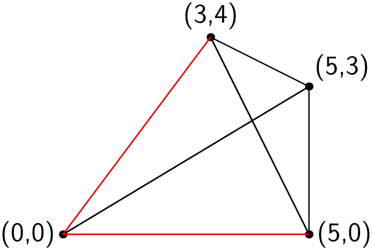
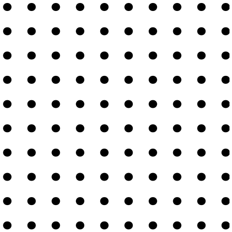
Counting distinct distances



A point set with 5 distinct distances: $\sqrt{5}, 3, 2\sqrt{5}, 5, \sqrt{34}$.



The $\sqrt{n} \times \sqrt{n}$ grid has about $\frac{n}{\sqrt{\log n}}$ distinct distances.

Conjecture (Erdős, 1946)

The minimum number of distinct distances spanned by n points is about $\frac{n}{\sqrt{\log n}}$.

Erdős distinct distances problem

Conjecture (Erdős, 1946)

The minimum number of distinct distances spanned by n points is about $\frac{n}{\sqrt{\log n}}$.

This conjecture was essentially resolved by Guth and Katz using incidence bounds:

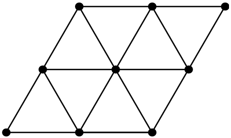
Theorem (Guth, Katz 2010)

The minimum number of distinct distances spanned by n points is at least $\frac{n}{\log n}$.

Conjecture (Erdős, 1946)

The minimum number of distinct distances spanned by n points is about $\frac{n}{\sqrt{\log n}}$.

However, Erdős also conjectured that the extremal construction "possesses lattice structure," and results in this vein have been much harder to prove.



One distinct distance

Question

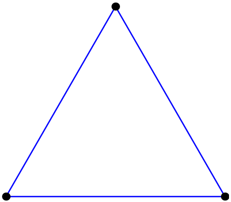
What is the maximum number of points in $\mathbb{R}^2$ that span one distance?

One distinct distance

Question

What is the maximum number of points in $\mathbb{R}^2$ that span one distance?

Answer: **3 points**.



Two distinct distances

Question

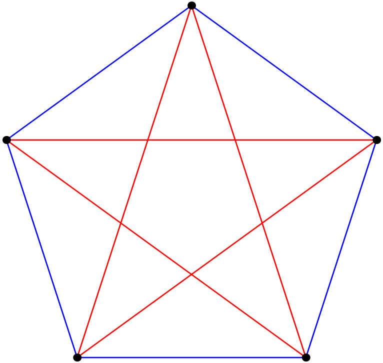
What is the maximum number of points in $\mathbb{R}^2$ that span exactly two distinct distances?

Two distinct distances

Question

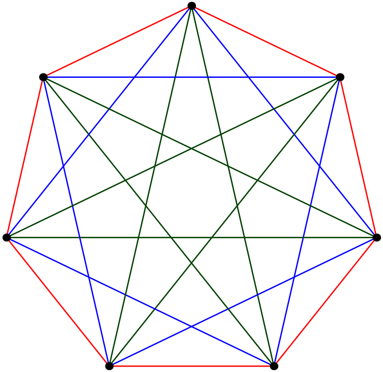
What is the maximum number of points in $\mathbb{R}^2$ that span exactly two distinct distances?

Answeer: **5 points**.



Erdős, Fishburn

For $k \in \mathbb{N}$, let $g(k)$ be the maximum number of points in $\mathbb{R}^2$ that span exactly k distances.

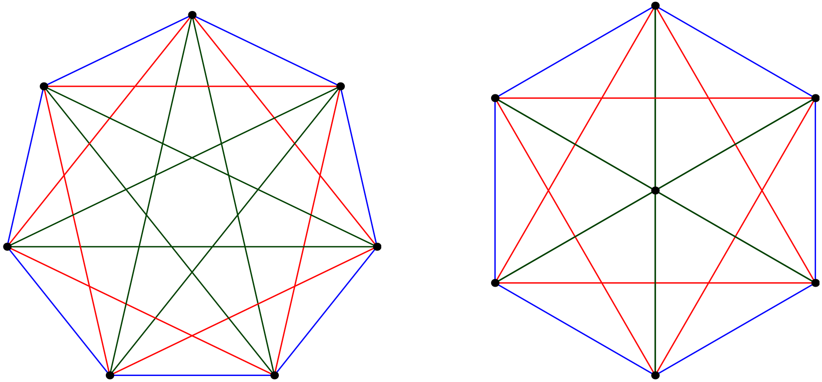


A regular $2k+1$ gon has k distinct distances, so $g(k) \geq 2k+1$ for all k.

Three Distinct Distances

Question

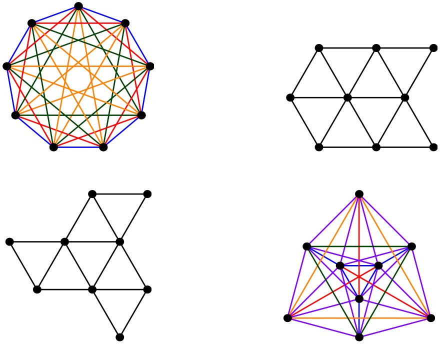
What is the maximum number of points in $\mathbb{R}^2$ that span exactly three distinct distances? (i.e. what is $g(3)$?)



$g(3) = 7$, and the unique extremal configurations are above.

QUESTION

What is the maximum number of points in $\mathbb{R}^2$ that span exactly four distinct distances?

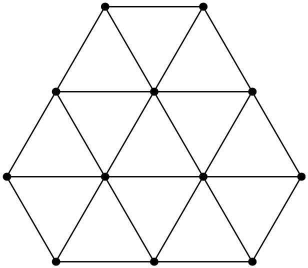


$g(4) = 9$, and the unique extremal configurations are above.

Question

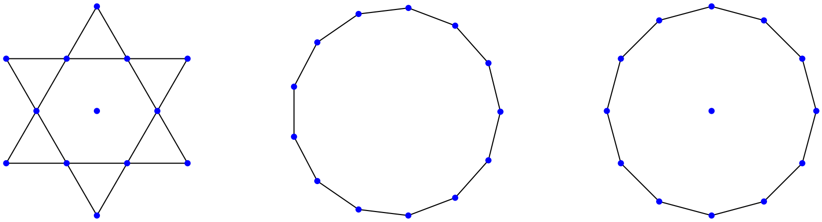
What is the maximum number of points in $\mathbb{R}^2$ that span exactly five distinct distances?

Answer: **12 points**.



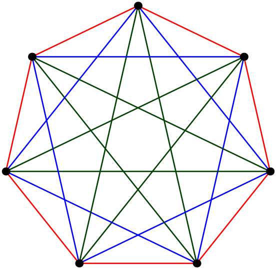
$g(5) = 12$, and the unique extremal configurations are above.

- Shinohara (2008) showed $g(6) = 13$.
- Open problem: what are the 13-point 6-distance sets?



The three such known point sets

Known general results



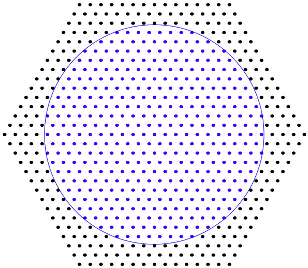
A regular $2k+1$ gon has k distinct distances, so $g(k) \geq 2k+1$.

Known general results

Triangular lattice bound (Balaji,Edwards,Loftin'2020) :

$$g(k) \geq 3\lfloor\sqrt{k+1}\rfloor^2 - 3\lfloor\sqrt{k+1}\rfloor + 1$$

From this, it can be easily shown that $g(k) > 2k+1$ for all $k > 6$.



A general construction for the above bound

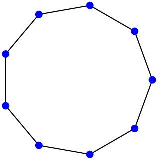
Overview of Known Results

k	g(k)	Completely Characterized?
1	3	Yes
2	5	Yes
3	7	Yes
4	9	Yes
5	12	Yes
6	13	No
7	≥ 16	No
8	≥ 19	No

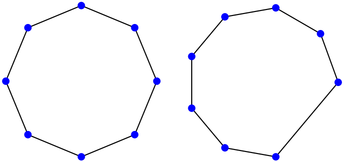
- **Main Goal:** Classifying all 13 point 6 distance sets.
- **Important Case:** Classifying 11-point 6-distance convex sets.
- **New Idea:** Computational technique to generate posets of distinct distances.
- **Similar Problem:** Progress in an adjacent problem about counting distinct triangles.

Lemma (Altman 1972)

Let P be a set of n points in convex position. Then P spans at least $\lfloor n/2 \rfloor$ distinct distances.



regular n-gon



regular n gon, and n vertices of regular $(n+1)$-gon

Casework on the 13 point 6 distance set

Goal: classify $g(6)$, the 13 point 6 distance sets.

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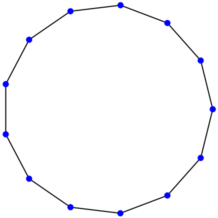
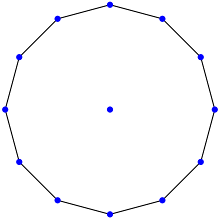
- 13 vertices on convex hull boundary:
- Altman Lemma: convex hull is regular 13-gon.



Only valid configuration for 13 points in convex hull

Goal: classify $g(6)$, the 13 point 6 distance sets.

- 12 vertices on convex hull boundary:
- Altman Lemma: convex hull is regular 12-gon, or 12 points of 13-gon.
- Only one way to add remaining point.

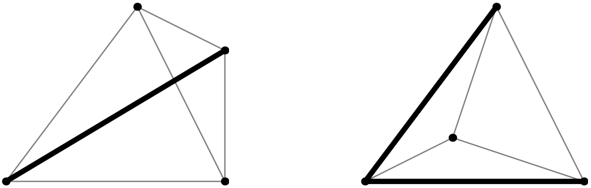


Only valid configuration for 12 points in convex hull

Diameters of point sets

Definition

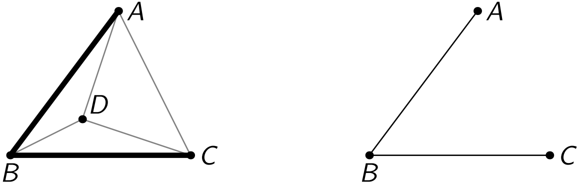
Given two points p, q of point set $P \in \mathbb{R}^2$, the segment pq is a **diameter** of P if no other segment is longer than pq.



Two point sets with diameter segments highlighted.

Definition

The **diameter graph** of $P \subset \mathbb{R}^2$ is the graph with vertex set on P whose edges are the diameters of P.

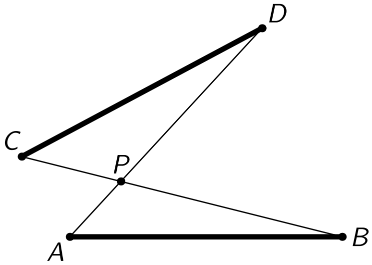


A point set and its diameter graph.

Diameter lemmas

Lemma

Any two diameters of a point set either intersect or share an endpoint.



Lemma

The set of endpoints of diameters is convex.

Convex 11 point 6 distance sets

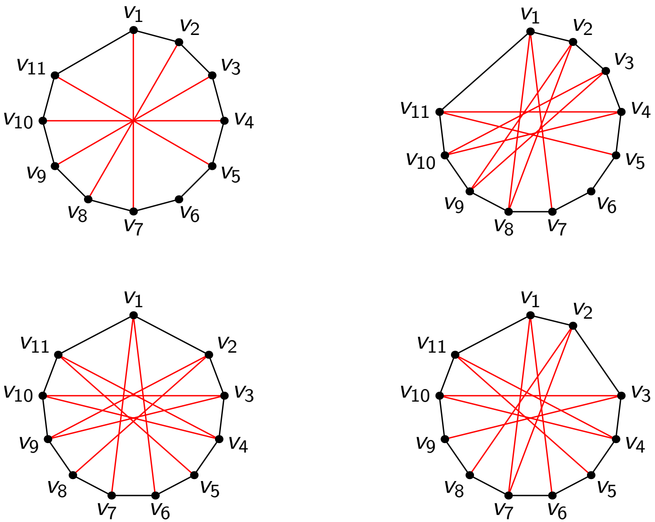
11 vertices on the convex hull boundary. . . significantly trickier!

Theorem (Main result)

If 11 points are in a convex position and span 6 distances, they must be either:

- *Any 11 points from the vertices of a regular 12-gon*
- *Any 11 points from the vertices of a regular 13-gon*

Convex 11 point 6 distance sets



4 of the 7 possible convex 11 point 6 distance sets

How was this classification attained?

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- Split into 3 big cases based on properties of the diameter graphs.

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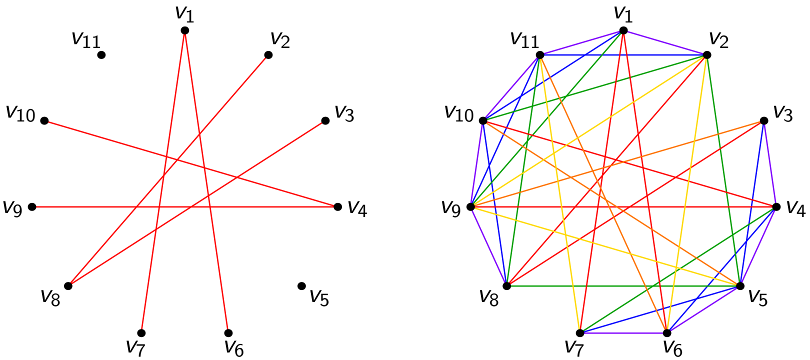
- Split into 3 big cases based on properties of the diameter graphs.
- Then, successively apply simple geometric techniques to deduce the remaining distances until a valid configuration, or a contradiction.

Example

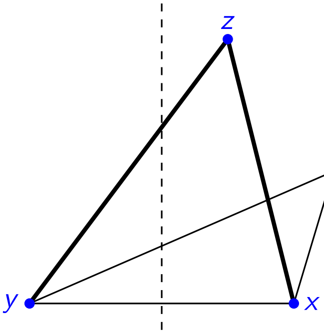


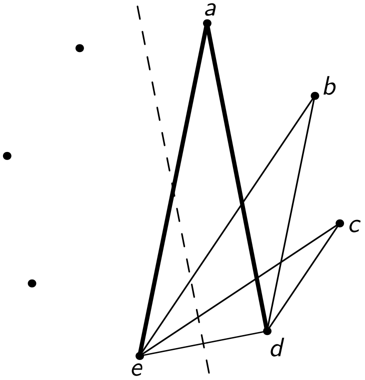
A diameter graph and the deduced distances in solving the case

Lemma (Perpendicular Bisector Lemma)

Let $x, y, z \in \mathbb{R}^2$, then:

$$d(x,z) < d(y,z) \iff z \text{ lies on the } x\text{-side of the perpendicular bisector of } xy$$





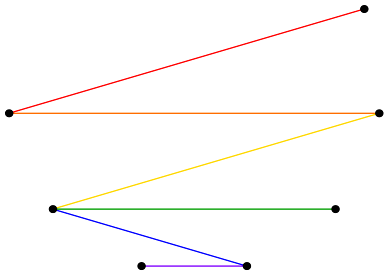
Long Chains of Distances

The perpendicular bisector lemma can yield chains of inequalities.

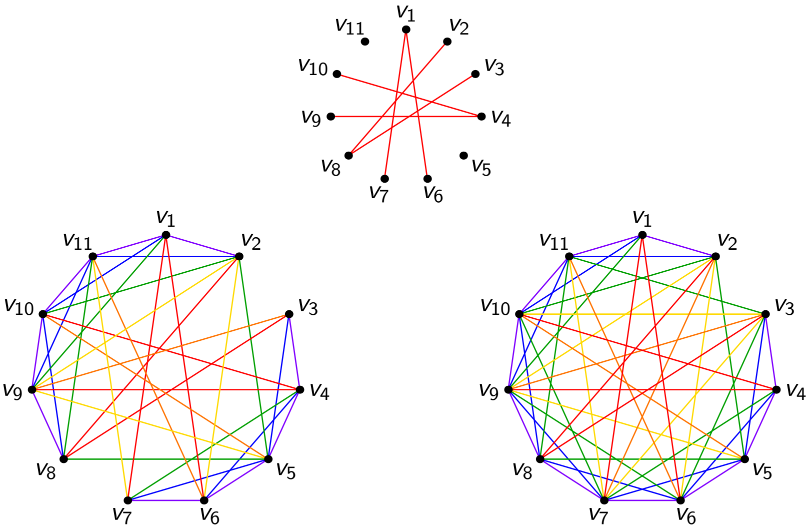
Lemma (Chain Lemma)

If we have line segments x_1x_2, x_3x_4,..., $x_{11}x_{12}$ in decreasing order and the possible distances are $d_1 > d_2 > \cdots > d_6$, then:

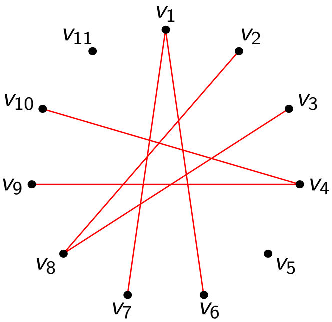
$$\begin{array}{ccccccc} d(x_1,x_2) & > & d(x_3,x_4) & > & \cdots & > & d(x_{11},x_{12}) \\ \| & & \| & & \ldots & & \| \\ d_1 & > & d_2 & > & \cdots & > & d_6 \end{array}$$



Computational Case Solving Tool



The diameter graph from before and the distances deduced by us (left) and by the program (right).



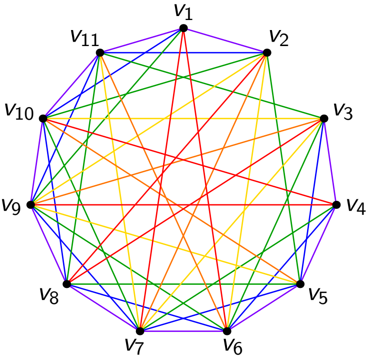

```
void initialDistances_N11_Case3AII
(ConvexDistanceGraph& cdg)
{
    assert(N == 11 && K == 6);
    // Set everything which is d1, and any
    other necessary edge inequalities.
    std::vector<std::pair<int, int>> d1_pairs =
    {{0, 5}, {0, 6}, {9, 3}, {9, 4}, {1, 7},
    {2, 7}}; // Note the zero-indexing
    std::vector<int> d1_edges;

    for(auto p : d1_pairs) d1_edges.push_back
    (cdg.lookup[p.first][p.second]);
    cdg.setD1Nodes(d1_edges, true); // It is
    assumed that there are no other edges of
    length d1
}
```

The diameter graph and how to encode it in the program

Computational Case Solving Tool: Result

Distance Table:

d	1	2	3	4	5	6	7	8	9	10	11
1	xx	6				1	1		4	5	6
2	6	xx			4	3	2	1	3	4	5
3			xx	6	5	4	3	1	2	3	4
4			6	xx	6	5	4		1	1	
5		4	5	6	xx	6	5	4	3	2	
6	1	3	4	5	6	xx	6	5	4		2
7	1	2	3	4	5	6	xx	6	5	4	3
8		1	1		4	5	6	xx	6	5	4
9	4	3	2	1	3	4	5	6	xx	6	5
10	5	4	3	1	2		4	5	6	xx	6
11	6	5	4			2	3	4	5	6	xx



The deduced edges in table and graphical form

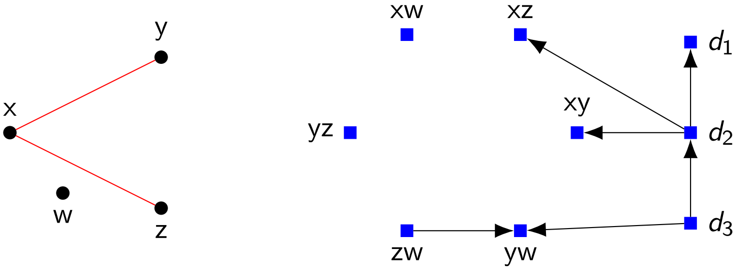
Overview of the Algorithm

```
repeat
    for all segments v_i v_j do
        updateLowerBound(v_i v_j)
        updateUpperBound(v_i v_j)
    end for
    applyBisectorLemma
    improveLowerBounds
    findChains
until no more new deductions are made
```

Our C++ code can also be found on GitHub:
`https://github.com/Puddlestomper/DistinctDistances`

Representing Edge Length Information

We represent the line segments and lengths in our sketch as the nodes in a directed graph, where an arrow from *ab* to *cd* indicates that $d(a, b) < d(c, d)$.



On the left we show initial conditions and on the right the corresponding representation in the program.

Main result

Theorem

If 11 points are in a convex position and span 6 distances, they must be either:

- *Any 11 points from the vertices of a regular 12-gon*
- *Any 11 points from the vertices of a regular 13-gon*

The following remains to be explored:

- Complete the classification of extremal point sets for $g(6)$;
- Compute the value of $g(7)$, the next open value;
- Improve computational tools to assist in the above;
- and more . . . .

Acknowledgements

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