List coloring with requests for planar graphs

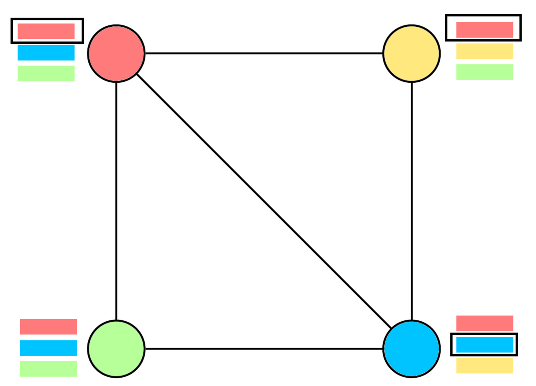
Fawzan Ali, Saurav Chittal, Siyu Gan, Yue Su
Bob Krueger, Peter Bradshaw

Introduction

Abstract
In 1994, Thomassen [3] proved that for every planar graph G with color lists of size five, there exists a coloring of G such that no two adjacent vertices share a color. Our goal is to show that with some additional property on G, we can always satisfy an ε-proportion of color requests, for some universal constant $\varepsilon > 0$.

Definitions
- A *planar graph* is a graph that can be drawn in the plane without any edge crossings.
- A *list assignment* for a graph G is a function that assigns each vertex $v \in V(G)$ a set $L(v)$ of colors, and an L*-coloring* is a proper coloring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$.
- A *request* for a graph G with a list assignment L is a function r with a domain $dom(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in dom(r)$.
- For $\varepsilon > 0$, we say that a request r is ε*-satisfiable* if there exists an L-coloring ϕ of G such that $\phi(v) = r(v)$ for at least $\varepsilon|dom(r)|$ vertices $v \in dom(r)$.
- A graph G with a list assignment L is called ε*-flexible* if every request is ε-satisfiable.
- For a given $\varepsilon > 0$, $n \in \mathbb{N}$, a graph G is ε*-flexibly* k*-choosable* if it is ε-flexible for every list assignment of lists of size k.
- The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident to v.
- The *maximum average degree* of a graph G, denoted mad(G), is the maximum of the average degree taken over all induced subgraphs of G.

Motivation
- Planar graphs with n vertices are $\frac{1}{n}$-flexibly 5-choosable [3].
- There exists $\varepsilon > 0$, such that mad$(G) < 4 + \frac{2}{5}$ implies G is ε-flexibly 5-choosable [2].
- If G is planar then mad$(G) < 6$.

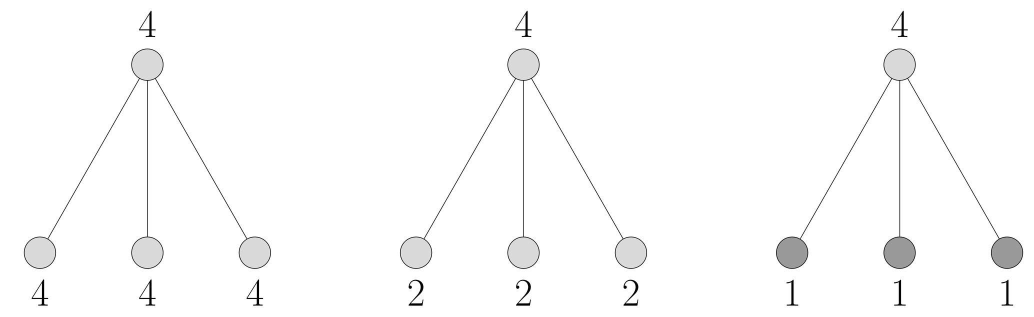


The graph on the left is an example of a list coloring with requests. Each vertex has a list of colors, with some lists having a color preference (represented with a black box).

Background

Definition: Reducible subgraph
Let G be a graph, let $H \subseteq G$ be an induced subgraph. For each $v \in V(H)$, define $\ell(v) = 5 - |N(v) \cap (G - H)|$. Then H is reducible if for every ℓ-assignment L on H, the following holds:
- FIX: $\forall v \in V(H)$, $\forall c \in L(v)$, there exist an L-coloring φ such that $\varphi(v) = c$.
- FORB: $\forall U \subseteq V(H)$ of size at most 3, $\forall c \in L(v)$, there exist an L-coloring φ of H such that $\varphi(u) \neq c$ for every $u \in U$.

Lemma (Dvořák, Masařík, Musílek, Pangrác [1]): If any induced subgraph of G has a reducible subgraph, then there exists $\varepsilon > 0$ such that G is ε-flexibly 5-choosable.



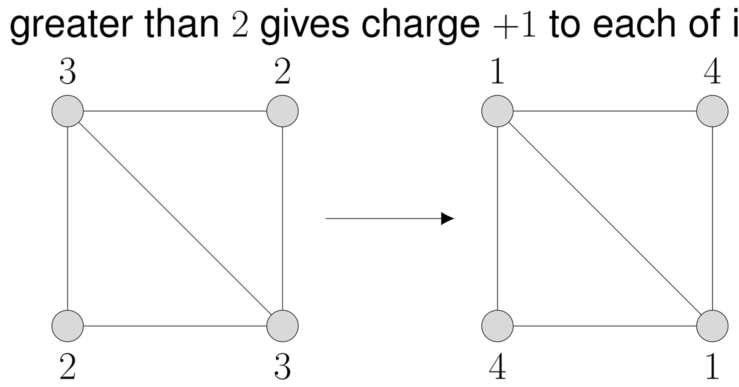
Pictured above is a reducible subgraph. On the left, the vertices are labelled with their degrees in G. In the middle, the vertices are labelled with their list sizes. On the right, the vertices are labelled with their list sizes after forbidding a color at the bottom three vertices (colored in dark gray).

Methods

Theorem ([2]): $\exists\varepsilon > 0$ such that if mad(G) < 4.4 then G is ε-flexibly 5-choosable.

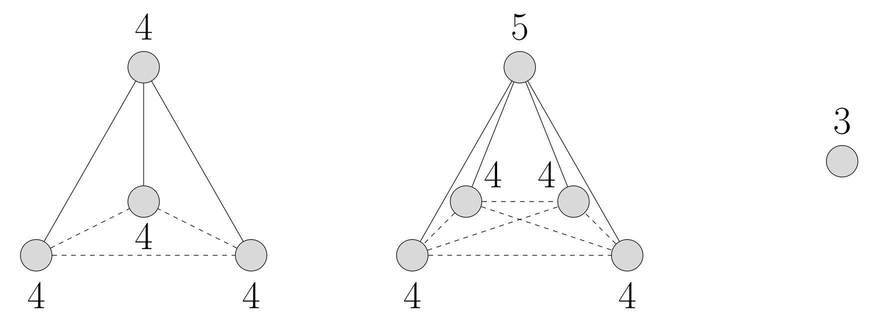
Discharging
- Assign initial charge to vertices and/or faces.
- Specify discharging rules, dictating how vertices/faces exchange charges. Total charge is conserved.
- Observe the final charges.

Example: Each vertex starts with charge equal to its degree. For the discharging rule, each vertex of degree greater than 2 gives charge $+1$ to each of its adjacent vertices.

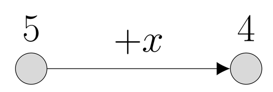


Goal: Assuming no reducible subgraphs, the sum of initial charges is negative, but the sum of final charges is positive, a contradiction.

Linear Programming
- We first establish that the following three graphs are reducible (dashed edge drawn indicates that there may be an edge present, label is degree of vertex in the graph):



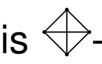
- **Goal:** Maximize m such that the following holds: if mad$(G) < m$, then one of the above structures appears in G.
- Initial charge at vertex v is $d(v) - m$. Note: sum of all initial charges is less than 0.
- Discharging rule: each vertex of degree at least 5 gives x charge to its neighbours of degree 4. Note: sum of all charges is conserved.

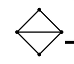


- For a contradiction, assuming that G has no reducible subgraph, we want the sum of final charges to be at least 0.
- Final charge of degree 4 vertex: $4 - m + 2x \geq 0$,
- Final charge of degree 5 vertex: $5 - m - 3x \geq 0$,
- Final charge of degree 6 vertex: $6 - m - 6x \geq 0$, etc.
- Set up the LP: maximize m such that there exists an x where the above constraints hold. The optimal solution to this LP is $(m, x) = (4.4, 0.2)$

Results

Theorem 1 (ABCGKS): There exists a universal $\varepsilon > 0$ such that if mad(G) $< 4 + \frac{16}{37}$ then G is ε-flexibly 5-choosable.

Theorem 2 (ABCGKS): There exists a universal $\varepsilon > 0$ such that if G is -free, and mad(G) $< 4 + \frac{4}{9}$ then G is ε-flexibly 5-choosable.

Theorem 3 (ABCGKS): There exists a universal $\varepsilon > 0$ such that if G is planar and -free then G is ε-flexibly 5-choosable.

Future Problems

Question 1
What is the maximum value m for which there exists $\varepsilon > 0$ such that mad(G) $< m$ implies that G is ε-flexibly 5-choosable?

Question 2
What is the maximum value m for which there exists $\varepsilon > 0$ such that mad(G) $< m$ implies that a K_4-free graph G is ε-flexibly 5-choosable?

Question 3
Could we create a database to store a large set of reducible graphs so that people can easily check whether a subgraph is reducible or not?

Conjecture [2]
There exists an $\varepsilon > 0$ such that all the planar graphs are ε-flexibly 5-choosable.

Acknowledgements

We would like to thank the IGL Program. IGL research is supported by the Department of Mathematics at the University of Illinois at Urbana-Champaign and by the National Science Foundation under Grant Number DMS-1449269. Peter Bradshaw was partially supported by by NSF RTG grant DMS-1937241.

References

[1] Z. Dvořák, T. Masařík, J. Musílek, and O. Pangrác. Flexibility of triangle-free planar graphs. *Journal of Graph Theory*, 96(4):619–641, 2021.

[2] Z. Dvořák, Sergey Norin, and Luke Postle. List coloring with requests. *Journal of Graph Theory*, 92(3):191–206, 2019.

[3] C. Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory*, Series B, 62(1):180–181, 1994.