Final Report

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1 Abstract

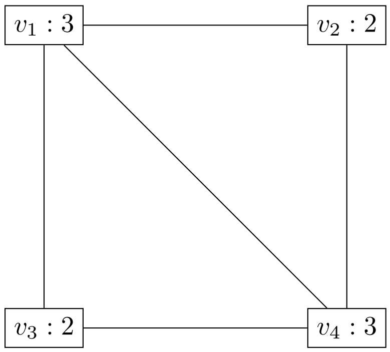
In 1994, Thomassen [3] proved that for every planar graph G with color lists of size five, there exists a coloring of G such that no two adjacent vertices share a color. Our goal is to show that with some additional property on G, we can always satisfy an ε-proportion of color requests, for some universal constant $\varepsilon > 0$.

2 Definitions

- A *planar graph* is a graph that can be drawn in the plane without any edge crossings.
- A *list assignment* for a graph G is a function that assigns each vertex $v \in V(G)$ a set $L(v)$ of colors, and an *L-coloring* is a proper coloring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$.
- A *request* for a graph G with a list assignment L is a function r with a domain $dom(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in dom(r)$.
- For $\varepsilon > 0$, we say that a request r is *ε-satisfiable* if there exists an L-coloring ϕ of G such that $\phi(v) = r(v)$ for at least $\varepsilon|dom(r)|$ vertices $v \in dom(r)$.
- A graph G with a list assignment L is called *ε-flexible* if every request is ε-satisfiable.
- For a given $\varepsilon > 0$, $n \in \mathbb{N}$, a graph G is *ε-flexibly k-choosable* if it is ε-flexible for every list assignment of lists of size k.
- The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident to v.
- The *maximum average degree* of a graph G, denoted $\text{mad}(G)$, is the maximum of the average degree taken over all induced subgraphs of G
- A 4-triangle a induced subgraph of G consist of a triangle made of 3 vertices of degree 4
- A 4-component is a component of G that only consists of vertices of degree 4
- Let G be a graph, let $H \subseteq G$ be an induced subgraph. For each $v \in V(H)$, define $\ell(v) = 5 - |N(v) \cap (G - H)|$. Then H is reducible if for every ℓ-assignment L on H, the following holds:
  - FIX: $\forall v \in V(H)$, $\forall c \in L(v)$, there exist an L-coloring φ such that $\varphi(v) = c$.
  - FORB: $\forall U \subseteq V(H)$ of size at most 3, $\forall c \in L(v)$, there exist an L-coloring φ of H such that $\varphi(u) \neq c$ for every $u \in U$.

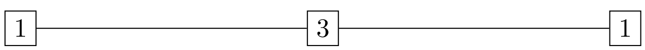
3 Lemma

Lemma 1: A graph like this is list colorable: The list size of the vertices is also exact the same as its degrees.



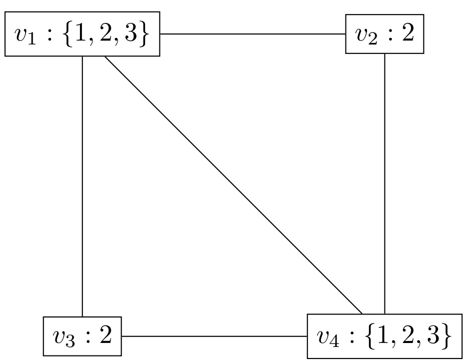
Proof:

First we examine v_1, suppose its lists are: $\{1, 2, 3\}$. Consider the lists of v_4, if it not the same as v_1, suppose 3 is not in its lists, we color vextex v_1 with color 1, and drop vertex v_1, the new graph becomes:

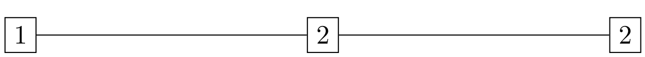


We can first color the two vertices with list size 1 and then color the vertex with list size 3, it is list colorable.

If the list of v_4 is the same as v_1



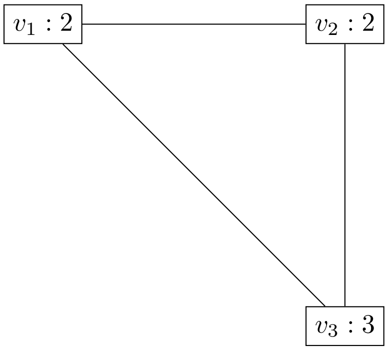
Consider the lists of v_2. If it has some color other than 1,2 and 3, suppose it is 4, and the rest color can be either 1,2, 3 or something else, but at least one of 1,2 and 3 are not included, suppose it is 1. Color v_1 with the color 1 and we drop v_1, the graph with list becomes:



We can color this graph from left to right, it is also list colorable.

If the lists of v_2 and v_3 are subsets of $\{1, 2, 3\}$, so v_2 and v_3 has at least one common color, suppose it is 1, and we can color: $v_1 = 2$, $v_2 = v_3 = 1$, $v_3 = 3$, it is also list colorable. ■

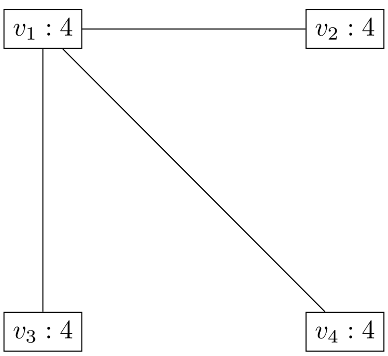
Lemma 2: A graph like this is also list colorable:



We can color v_1 first, and then color v_2, then v_3, it is list-colorable. ■

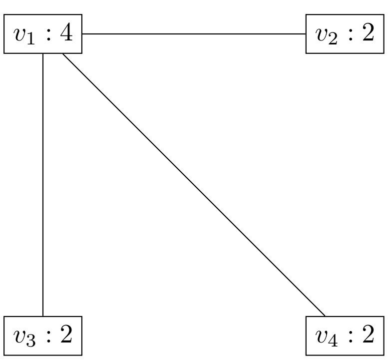
Lemma 3: Any vertex of degree less or equal to 3 is a reducible subgraph.
Proof: Suppose the degree is d, $d \leq 3$, $5 - d \geq 2$. For Fix, obviously it holds. For Forb, since it has at least two colors, Forb also holds. So it is a reducible subgraph.■

Lemma 4: If any vertex of degree 4 who has more than 2 neighbours of degree 4, there is a reducible subgraph.

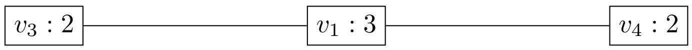


There are several soncitions between them and we need to consider them one by one

Condition 1: There is no edge between v_2 and v_3, v_2 and v_4, v_3 and v_4. To prove reducibility, we just need to prove that this graph with lists is list-colorable:

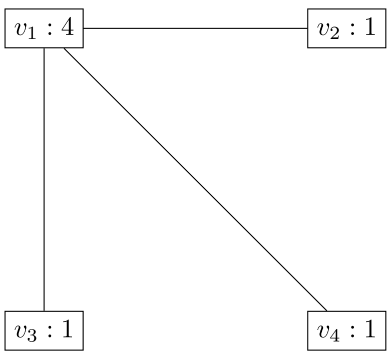


Check Fix: If we fix v_1, all the other 3 vertices has at least one color to choose, it is colorable. If we choose v_2 oe v_3 or v_4, suppose it is v_2, we drop v_2, and the graph with lists becomes:



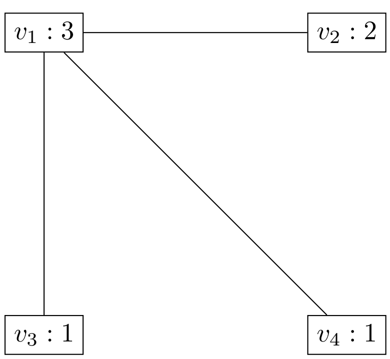
Color it from left to right, it is list-colorable.

Check Forb: If we forb all the 3 vertices of degree 4, the graph with lists becomes:



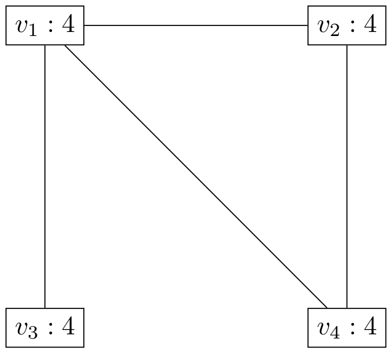
We first color v_2, v_3 and v_4, and v_1 has at least one color to choose, it is list-colorable

If we forbid v_1 and two of the other vertices, suppose they are v_1, v_4 and v_3, the graph becomes:

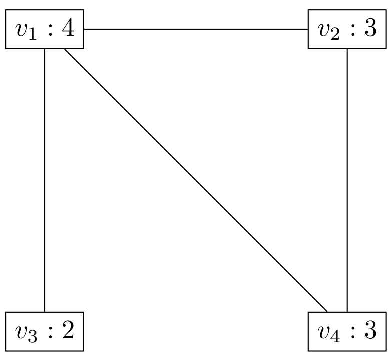


First we color v_4 and v_3, v_1 has at least one color to choose, we color v_1 and then v_2, it is also list-colorable.

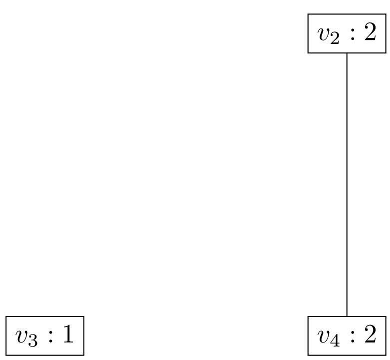
Condition 2: there is exactly one edge between v_2 and v_3, v_2 and v_4, v_3 and v_4. Suppose it is v_2 and v_4, the graph becomes:



To prove it is reducible, we just need to prove that this graph is list colorable:

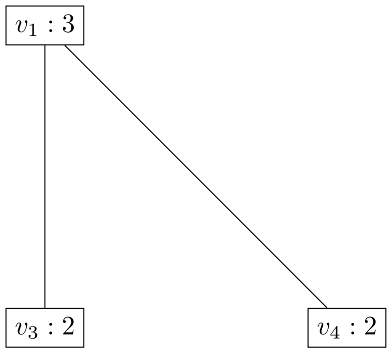


Check Fix: If we fix v_1, we drop v_1 and the graph becomes



It is trivial that it is list-colorable.

If we fix v_2 or v_4, suppose it is v_2(the other is the same) we drop v_2, and the graph becomes:

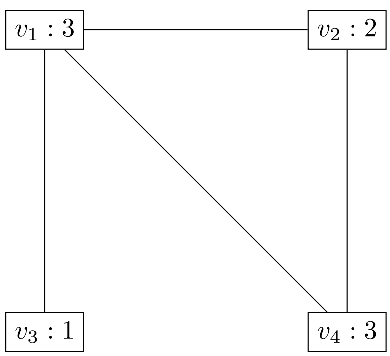


It is trivial that it is list-colorable.

If we fix v_3, by lemma 2, it is also list-colorable.

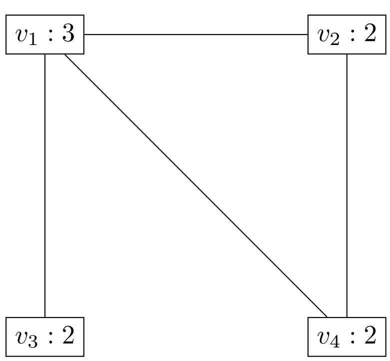
Check Forb:

If we fix v_1, v_2 and v_3, the graph becomes:



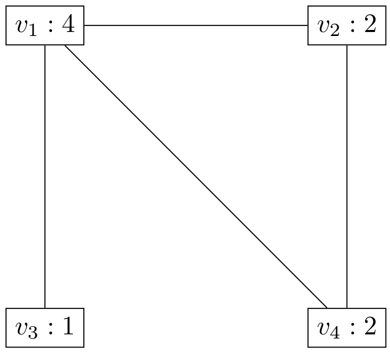
We color v_3 first and drop it, by lemma 2, it is list colorable.

If we fix v_1, v_4 and v_2, the graph with lists becomes:



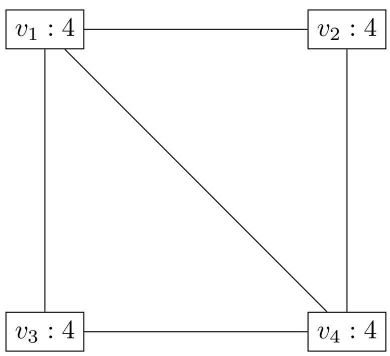
We first color v_2 and v_4, and then v_1, v_3, it is also list-colorable

If we forb v_3, v_2 and v_4, the graph with lists becomes:

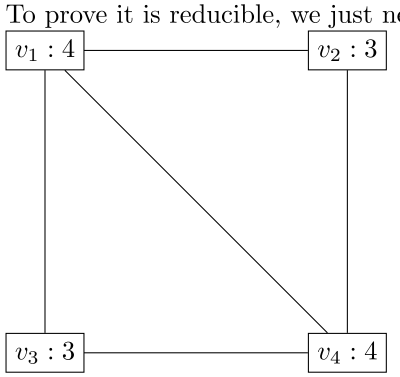


First we color v_3 and drop it, by lemma 2, it is list colorable.

Condition 3: there are exactly two edges between v_2 and v_3, v_2 and v_4, v_3 and v_4. Suppose it is v_2 and v_4, v_3 and v_4, the graph becomes:

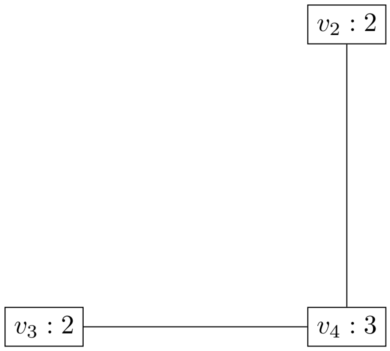


To prove it is reducible, we just need to prove that this graph is list-colorable:



Check Fix:

If we fix v_1 or v_4, suppose we fix v_1, the graph becomes:

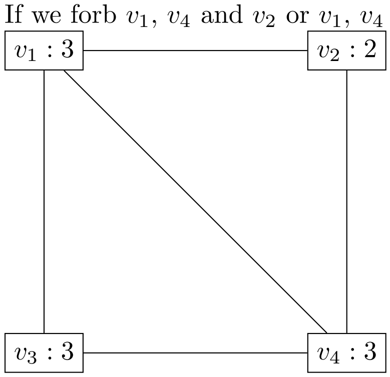


It is trivial that this graph with lists is list-colorable.

If we fix v_2 or v_3, we drop it and by lemma 2, it is still list-colorable.

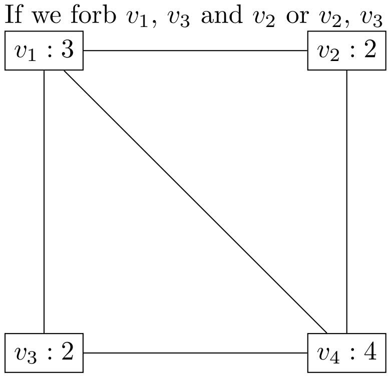
Check forb:

If we forb v_1, v_4 and v_2 or v_1, v_4 and v_3, suppose it is v_1, v_4 and v_2, the graph becomes:



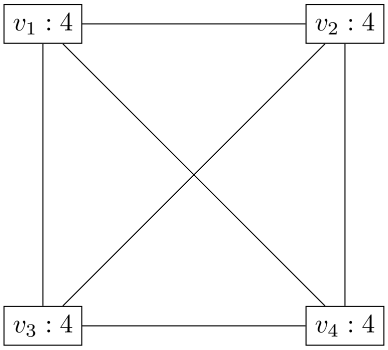
First we color v_2 and drop it, and by lemma 2, it is list colorable.

If we forb v_1, v_3 and v_2 or v_2, v_3 and v_4, suppose we ford v_1, v_3 and v_2, the graph with lists becomes:

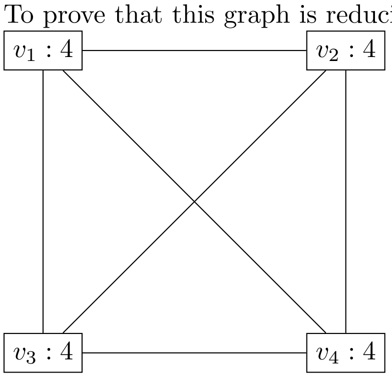


First we color v_3, and by lemma 2, it is still list-colorable.

Condition 4: If vertices v_2, v_3, v_4 are adjacent to one another, the graph becomes:

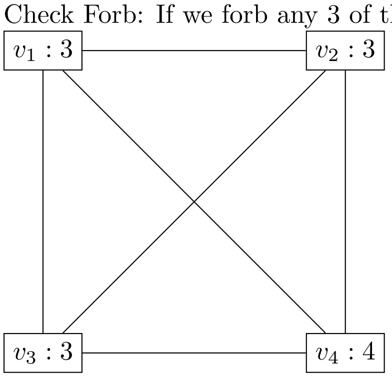


To prove that this graph is reducible, we need to prove that this graph is list-colorable:



Check Fix: If we fix any one of the vertex, we drop it, and by lemma 2, it is list-colorable.

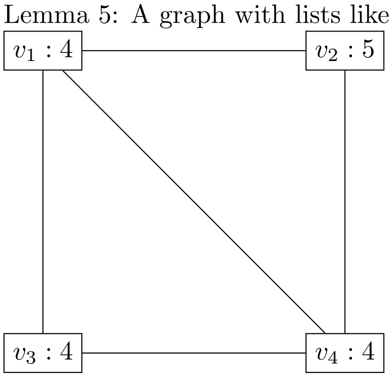
Check Forb: If we forb any 3 of the vertices, suppose they are v_1, v_2 and v_3, the graph with lists become:



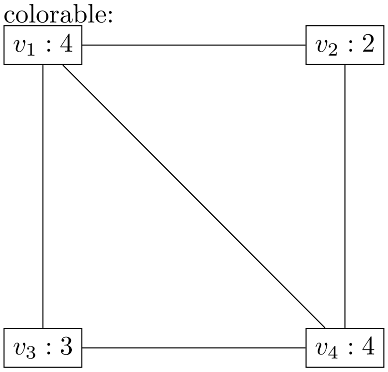
We first color v_4 and drop it, and by lemma 2, the graph is list-colorable.

Judging all those cases, v_1, v_2, v_3 and v_1 form a reducible subgraph. ■

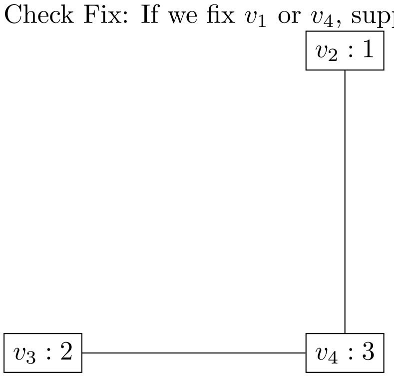
Lemma 5: A graph with lists like this is reducible:



To prove that this graph with lists is reducible, we need to prove that this graph below with lists is list-colorable:



Check Fix: If we fix v_1 or v_4, suppose we fix v_1, we drop it and the graph with lists becomes:

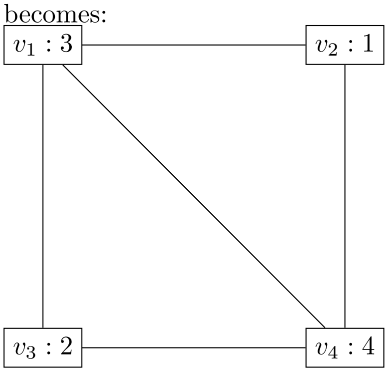


It is trivial that this graph is list-colorable

If we fix v_2 and drop it, by lemma 2, it is list-clorable

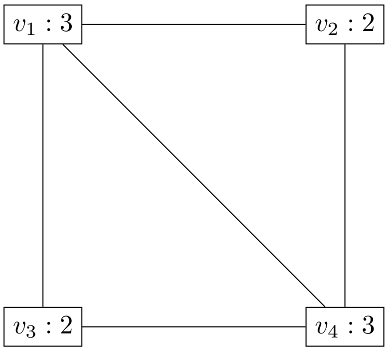
If we fix v_3 and drop it, by lemma 2, it is list-colorable

Check Forb: If we forb v_1, v_2 and v_3 or v_4, v_2 and v_3, suppose we forb v_1, v_2 and v_3, the graph with lists becomes:



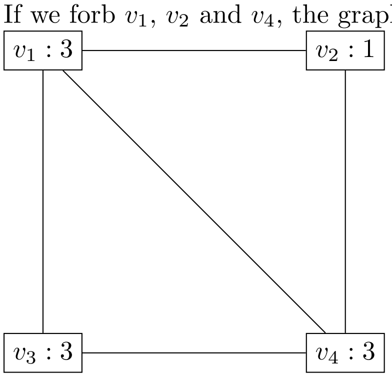
We first color v_2 and drop it, by lemma 2, it is list-colorable

If we forb v_1, v_3 and v_4, the graph with lists becomes:



By lemma 1, it is still list-colorable.

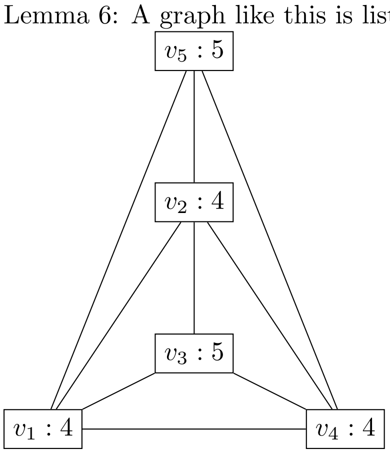
If we forb v_1, v_2 and v_4, the graph with lists becomes:



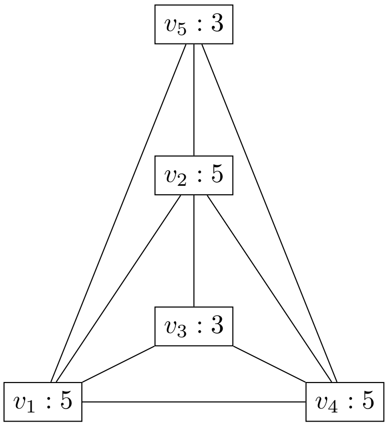
We first color v_2 and drop it. By lemma 2, it is list-colorable.

Juding all these cases, the graph is reducible ■

Lemma 6: A graph like this is list colorable:

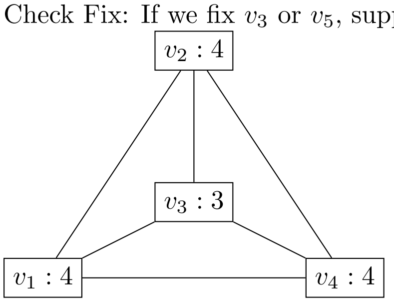


To prove it, we need to prove that this graph with lists is list-colorable:



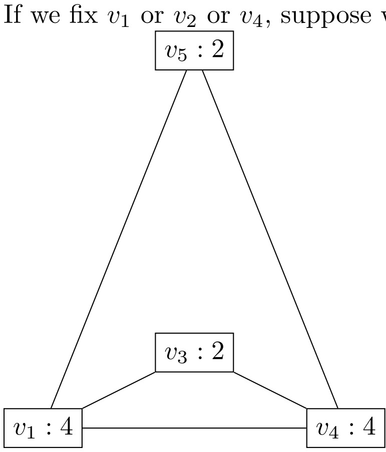
To prove it, we need to prove that this graph with lists is list-colorable:

Check Fix: If we fix v_3 or v_5, suppose we fix v_5, and drop it, the graph becomes:



Color v_3 and drop it, by lemma 2, it is list-colorable

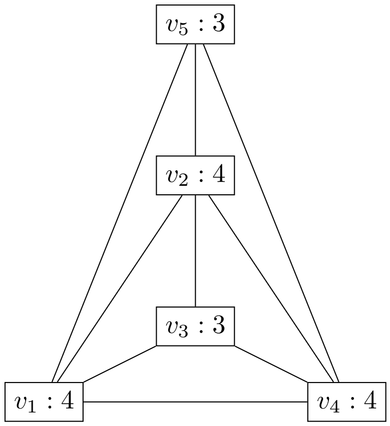
If we fix v_1 or v_2 or v_4, suppose we fix v_2 and drop it, the graph becomes:



Color v_5 and drop it, by lemma 2, it is list-colorable.

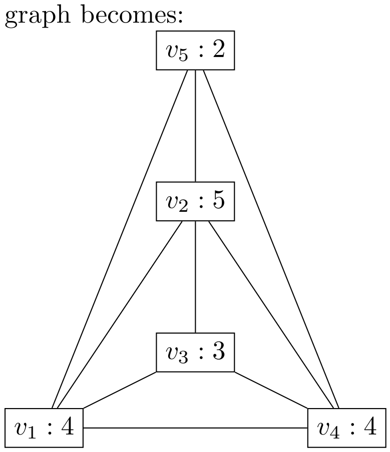
Check Forb: There are three conditions: forbid 0,1,or 2 vertices of list size 3.

Condition 1: If we forbid no vertex of list size 3, which means we forbid v_1, v_2 and v_4. The graph becomes:

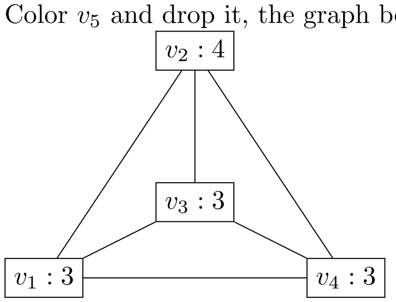


Color v_1 and drop it, by lemma 1, it is list-colorable.

Condition 2: If we forb exactly one of the vertices of list size 3, which means we forb v_1, v_4 and v_5. The graph becomes:



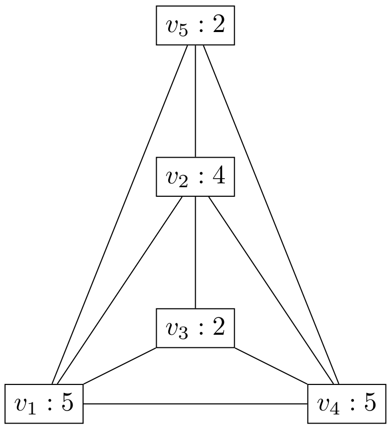
Color v_5 and drop it, the graph becomes:



Color v_1 and drop it, by lemma 2, it is list colorable.

Condition 3: If we forb two vertices of list size 3, suppose it is v_2, v_3 and v_5, the graph becomes:

Condition 2: If we forb exactly one of the vertices of list size 3, which means we forb v_1, v_4 and v_5. The graph becomes:



Color v_3 and v_5 and drop them respectively, by lemma 2, the graph is list colorable.

Judging all the cases, the graph is reducible. ■

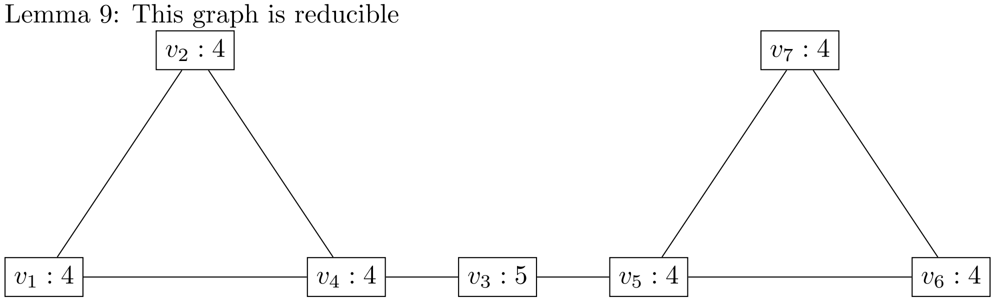
Lemma 7: For any vertex of degree 5, if is has more than 3 neighbours of degree 4, the graph has a reducible subgraph. (This lemma is proven by computers)

Lemma 8: every 4-component without a circle and with more than 4 vertices is reducible. (This lemma is proven by computers)

Corollary of Lemma 8: For every vertex v in a 4-component, suppose the number of its neighbours with degree greater than 4 is $n(v)$, the average $n(v)$ for each of the component is at least $\frac{8}{3}$.

Proof of the corollary: suppose the component consists of k vertices, in which $k \leq 3$ and since it is connected and it has no edges, it is a spannign tree with $n-1$ edges. So it has $n-1$ edges inside the component, and it has $4 \cdot n - 2 \cdot (n-1) = 2n+2$ neighbours in all. The average $n(v) = \frac{2n+2}{n} = 2 + \frac{2}{n} \geq 2 + \frac{2}{3} = \frac{8}{3}$

Lemma 9: This graph is reducible



This lemma is also proven with the help of computers.

Lemma 10: (Dvořák, Masařík, Musílek, Pangrác [1]): If any induced subgraph of G has a reducible subgraph, then there exists $\varepsilon > 0$ such that G is ε-flexibly 5-choosable.

4 Theorems

Theorem 1: For a planar graph G, if $\text{mad}(G) < 4 + \frac{16}{37}$, there is an $\varepsilon > 0$, such that the graph is $\varepsilon-$flexibly 5$-$choosable.

Proof: If there is a reducible subgraph in any of the subgraphs, by lemma 10, there is a $\epsilon > 0$, such that the graph is $\epsilon-$flexibly 5$-$choosable. So we just need to prove by contradiction and assume that there is not a reducible subgraph in the graph.

We set up these initial charges:

According to lemma 3, all vertices of degree less or equal to 3 is a reducible subgraph, so we just need to consider about vertices of degree greater or equal to 4. For each vertex v, suppose its degree is $deg(v)$, its initial charge is: $deg(v) - 4 + \frac{16}{37}$. Notice that since $mad(G) < 4 + \frac{16}{37}$, the average degree of the graph is also less than $4 + \frac{16}{37}$, so the sum of all the initial charge is less than 0. (1)

We then set up these discharging rules: For each vertex of degree 5, give $x = \frac{6}{37}$ to all its neighbours of degree 4, which is not in a 4-triangle

For each vertex of degree 5, if it is adjacent to all the 3 vertices in a 4-triangle, give $w = \frac{7}{37}$ to all of its neighbours in a 4-triangle.

For each vertex of degree 5, if it is adjacent to exactly one of the vertices in a 4-triangle, give $y = \frac{9}{37}$ to this neighbour.

For each vertex of degree 6 or more, give $z = \frac{9}{37}$ to all its neighbours of degree 4, regardless it is a vertex in a 4-triangle or not And finally, for each 4-component, average all the charges on each vertex of the component

Now lets consider about what the final charges will become:

First, for vertices of degree 5, according to lemma 7, it can have at most 3 neighbours of degree 4. If it is adjacent to 3 vertices in a 4-triangle, its final charge will become: $5 - (4 + \frac{16}{37}) - \frac{7}{37} \cdot = 0 \geq 0$. And according to lemma 5, it can not be adjacent to exactly 2 neighbours of degree 4 in a 4-triangle, otherwise there will be a reducible subgraph. According to lemma 9, if it is adjacent to exactly one neighbour of degree 4 in another 4-triangle, it can not be adjacent to another neighbour of degree 4 in another 4-triangle, otherwise there will be a reducible subgraph. So it can have at most one neighbour of degree 4 in a 4-triangle, and 2 neighbours of degree 4 not in a 4-triangle. Its final charge will be at least: $5 - (4 + \frac{16}{37}) - \frac{9}{37} - \frac{6}{37} \cdot 2 = 0 \geq 0$. If it is not adjacent to any of vertex of degree 4 in a 4-triangle, it can have at most 3 neighbour not in a 4-triangle, its final charge will be at least $5 - (4 + \frac{16}{37}) - \frac{6}{37} \cdot 3 = \frac{3}{37} > 0$. Judging all the cases, the final charges vertices of degree 5 will be non-positive.

Second, consider the final charges of the vertices in the 4-triangles: According to lemma 4, a vertex of degree 4 can have at most 2 neighbours of degree 4. This means that a 4-triangle can receive exactly $3 \cdot (4 - 2) = 6$ charges from its neighbours, since a vertex of degree 4 has already had 2 neighbours of degree 4 in the 4-triangle. According to our discharging rules, these charges have to be either $\frac{7}{37}$ or $\frac{9}{37}$. But according to lemma 6, these 6 charges can not all be $\frac{7}{37}$, otherwise there will be a reducible subgraph. So the sum of final charges of the 4-triangle will be at least $3 \cdot (4 - (4 + \frac{16}{37})) + 3 \cdot \frac{7}{37} + 3 \cdot \frac{9}{37} = 0$, which is also non-negative

Third, consider the final charges of vertices of degree 4 not in a 4-triangle. According to lemma 8 and its corollary, the minimum charge a 4 vertex not in a 4-triangle receive will be $\frac{8}{3}x$, so the final charge will be greater or equal to $4 - (4 + \frac{16}{37}) + \frac{8}{3} \cdot \frac{6}{37} = 0$, which is also non-negative

Finally, for vertices of degree greater or equal to 6, suppose its degree is d, $d \geq 6$ and it can give away at most d charges to vertices of degree 4. So its final charge will be greater equal to $d - (4 + \frac{16}{37}) - d\frac{9}{37} = \frac{28}{37}d - (4 + \frac{16}{37}) \geq 6 \cdot \frac{28}{37} - (4 + \frac{16}{37}) = \frac{4}{37} > 0$, which is also non-negative

Judging all those cases, the final charges of every vertex in the graph is non-negative, which is against our statement (1). This causes a conflict, and there should be a reducible subgraph in the graph, hence there is a $\varepsilon > 0$, such that the graph is $\varepsilon-$flexibly $5-$choosable. ■

Theorem 2: If a planar graph G has no adjacent triangle faces (i.e., no $K_4 - e$ and no bowtie graph as induced subgraphs), then G is ε-flexibly 5-choosable.

Proof: We show that G must have a reducible subgraph. BWOC, suppose G has no reducible subgraph. For every vertex $v \in V(G)$, we assign charge $d(v) - 4$ to v. For every face f of G, we assign $d(f) - 4$ charge to f. By Euler's formula, the sum of charges is -8. Since $\delta(G) \geq 4$ (otherwise, we may always find a reducible subgraph), the only negative charges are the triangle faces. Since we have forbidden $K_4 - e$ as an induced subgraph, we are in a K_4-free setting, and therefore a triangle where all vertices have degree 4 is reducible.

As a result, each triangle must have a vertex of degree at least 5. Such a vertex has charge greater or equal to 1, so it can give a charge of 1 to its adjacent triangle face. This means the sum of all charges is non-negative. Since the discharging rule does not change the overall charge, this is a contradiction. ■

Theorem 3: For a planar graph G, if there is no K_4 subgraph and $\text{mad}(G) < 4 + \frac{80}{139}$, then there is an $\varepsilon > 0$, such that the graph is $\varepsilon-$flexibly $5-$choosable.

Proof: Since G is K_4-free, by the lemma, it cannot have a triangle with all vertices degree 4. So let we define some cases in discharging rules:

- Let x be degree 5 given to degree 4 in a triangle, where degree 4 vertices can form a P_3
- Let y be degree 5 given to degree 4 not in a triangle, where degree 4 vertices can form a P_3.
- Let v be degree 5 given to degree 4 in triangle, where degree 4 vertices can form a P_2.
- Let w be degree 5 given to degree 4 not in triangle, where degree 4 vertices can form a P_2.
- Let u be degree 5 given to degree 4, where this degree 4 does not connect any other degree 4 vertex.

Using the $\text{mad}(G) < m$, we will get

- $x = \frac{30}{139}$;
- $y = \frac{30}{139}$;
- $w = \frac{29}{139}$;
- $u = \frac{15}{139}$;
- $v = \frac{22}{139}$;

So we have our discharging methods, for degree 6 or more, it can connect one or more vertices of degree 4. So it will give exactly one charge to that degree.

5 References

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