Robustness to Manipulation in Voting Theory

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- Judge systems based on their properties.
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- Robustness to manipulation: which systems lead to the fewest number of elections an adversary could manipulate to change the results?
- Techniques: simulations and proofs

Two Candidates

Election Terminology:

- **ballot:** 0 or 1
- **election:** n voters choosing 0 or 1, $\{0,1\}^n$
- **voting system:** function $f : \{0,1\}^n \to \{0,1\}$
- **balanced voting system:** equal chances of each candidate being the winner
- **majority:** voting system for odd n where candidate with most votes wins
- **leave-one-out majority:** for even n, disregarding the vote of one voter across all elections to avoid ties
- ***t*-manipulable election:** changing at most t ballots can result in a different winner

Two Candidates

Of all 2-candidate n-voter voting systems, which one minimizes the number of t-manipulable elections?

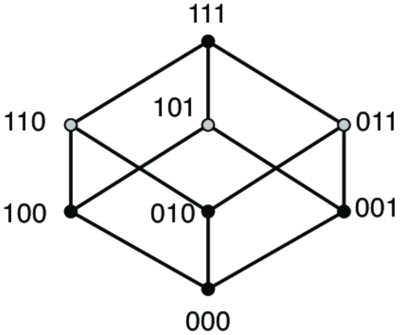
Theorem

For odd n, majority minimizes the number of t-manipulable elections. (Heilman '20)
For even n, leave-one-out majority minimizes the number of t-manipulable elections.

Two Candidates

Main tool: Harper's Theorem

- **hypercube:** n-dimensional square/cube: vertices $\{0,1\}^n$, 0-1 vectors of length n, where two vertices are adjacent if they differ in exactly one coordinate.



Two Candidates

Main tool: Harper's Theorem

- **hypercube:** n-dimensional square/cube: vertices $\{0, 1\}^n$, 0-1 vectors of length n, where two vertices are adjacent if they differ in exactly one coordinate.
- **lexicographic order:** alphabetical order of vertices
- **simplicial order:** ordering that first orders by the number of zeros/ones, then within those sets of vertices, orders lexicographically
- **boundary of S:** vertices not in S with a neighbor in S

Two Candidates

Theorem (Harper '66)

For every ℓ, a subset S of size ℓ of the hypercube of minimum boundary is given by an initial segment of simplicial order.

Simplicial order - Majority: $n = 3$

- 111
- $011, 101, 110$
- $001, 010, 100$
- 000

Simplicial order - Leave-one-out Majority: $n = 4$

- 1111
- $0111, 1011, 1101, 1110$
- $0011, 0101, 0110, 1001, 1010, 1100$
- $0001, 0010, 0100, 1000$
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Two Candidates

Theorem (Harper '66)

For every ℓ, a subset S of size ℓ of the hypercube of minimum boundary is given by an initial segment of simplicial order.

- vertices = elections
- subset of hypercube S = elections where candidate 1 wins
- boundary = manipulable elections
- initial segment of simplicial order = majority or leave-one-out majority

Simulations: Terminology

λ-Borda count voting system:

- voters rank every candidate
- assign points to each candidate: 1 to 1st, λ to 2nd, 0 to 3rd
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	1pt		$\frac{1}{2}$pt		0pt
5:	A	$>$	B	$>$	C
3:	B	$>$	C	$>$	A
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A:	5(1)	+	3(0)	+	1(0)	=	5
B:	5(1/2)	+	3(1)	+	1(1/2)	=	6
C:	5(0)	+	3(1/2)	+	1(1)	=	3.5

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B is selected as the winner.

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λ-Borda elimination voting system:

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- eliminate lowest scoring candidate
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C is eliminated.

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	1pt		0pt
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$$A: \quad 5(1) + 3(0) + 1(0) = 5$$
$$B: \quad 5(0) + 3(1) + 1(1) = 4$$

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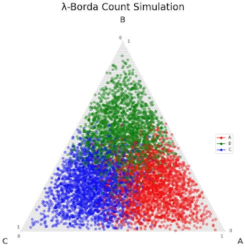
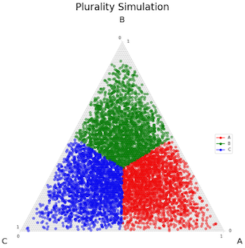
$$A: \quad 5(1) + 3(0) + 1(0) = 5$$
$$B: \quad 5(0) + 3(1) + 1(1) = 4$$

B is eliminated; A wins.

Simulations: Terminology

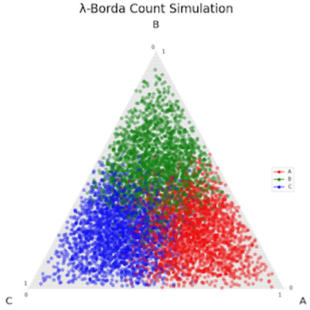
- These systems can be visualized in an **equilateral triangle**
- **Vertex** represents one candidate receiving all top-place votes
- **Distance to side**: proportion of top-place votes that each candidate received
- **Color** determined by winner

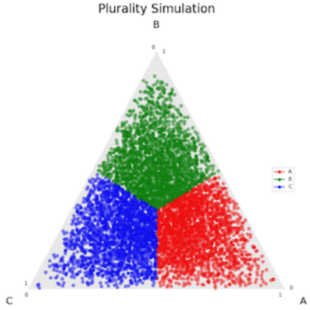




Simulations: Terminology

- **Manipulability**: proportion of elections where we can change at most ϵ-proportion of each type of ballot to change the outcome of the election

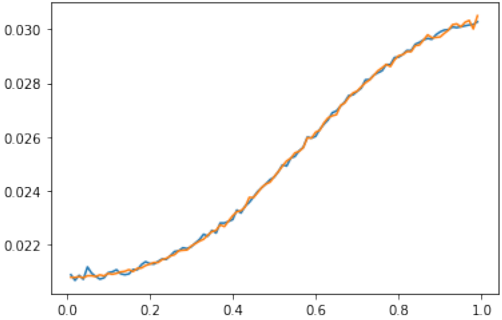




Polytope Calculation

- **Polytope**: set of all points in $\mathbb{R}^6$ satisfying some linear inequalities
  - generalized as a polyhedron in any dimension
  - enclosed by hyperplanes represented by linear inequalities
  - linear inequalities derived from x_1, x_2, x_3, x_4, x_5, and x_6
- Manipulability = volume of the boundary between regions with different winners
- Finding the polytope volume determines the manipulability at a given λ

Polytope Volume with respect to λ



- Lower $\lambda \rightarrow$ lower polytope volume $\rightarrow$ lower manipulability
- Plurality voting system (λ = 0) has the least manipulability

Generalized Borda Count

- **Simulations**: using python to graphically display how manipulability changes based on λ.
  - **Input**: 6 numbers, each representing a proportion of voters for a ballot.
  - **Manipulability**: proportion of elections manipulated by changing ϵ-proportion of ballots
  - **Perturbation**: changing ϵ-proportion of the ballots to check if manipulable

Manipulability and Perturbations

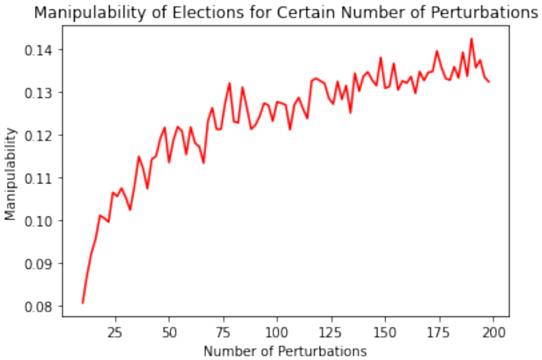


Figure: Finding the necessary number of perturbations for accuracy and efficiency.

Manipulability vs Epsilon

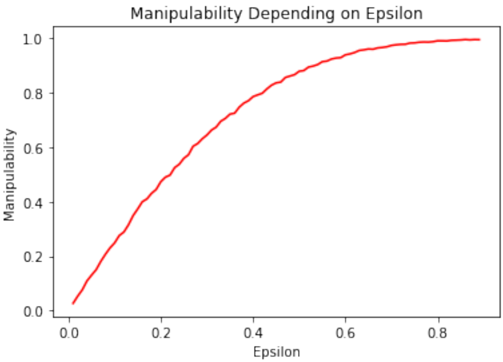
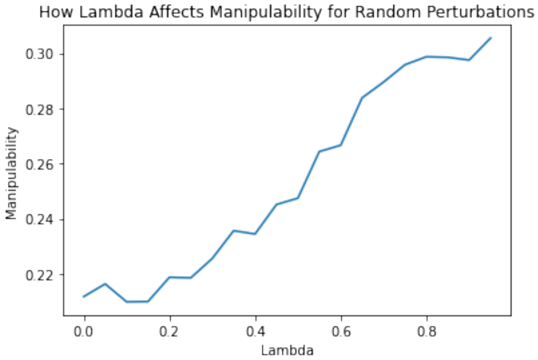


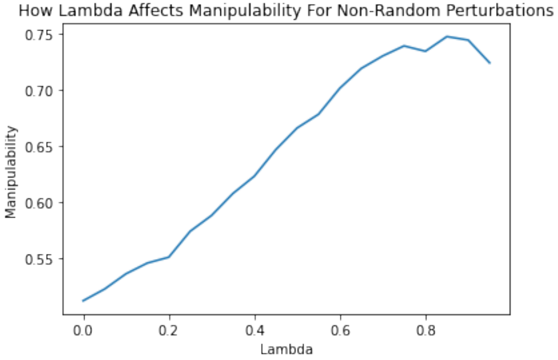
Figure: Finding a realistic and effective epsilon for the simulations.

Manipulability vs. Lambda Random Perturbations

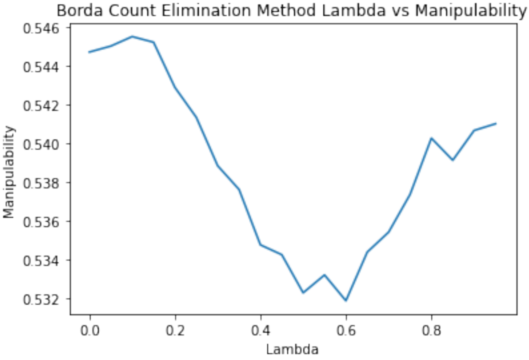


- Lower Lambda, Lower Manipulability
- Lambda of 0 corresponds to Plurality

Manipulability vs. Lambda Non-Random Perturbations



Generalized Borda Elimination



Open Questions and Future Work

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We thank David Frankel (Uni High class of 1976) whose gift made this experience possible for University Laboratory High School students.

Thanks for listening.